Motivating High-Growth and Measuring Its Risk: A Dynamic Contract Approach*

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Abstract

We introduce the implicit probability of growth: a forward looking quantity that measures the relative risk priced in by investors upon making an investment. In order to achieve it, we derive the optimal dynamic contract in a continuous-time, principal-agent model tailored to accommodate key determinants of early-stage firms. The entrepreneur can either implement an existing technology that performs well in the near-term with little or no future growth or a new high-growth technology that performs poorly in the near-term with significant future growth. In the optimal contract, motivating the entrepreneur to pursue high-growth requires rewarding after histories of poor performance. The contract is implemented by opening a limited liability firm jointly owned by investors and the entrepreneur.

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1 Motivation

The risk associated with early-stage, start-up investments is difficult to measure; these firms are not publicly traded, data is hard to come by and observed returns suffer a significant selection biased. Even more so, in their recent work Gompers, Gornall, Kaplan, and Strebulaev (2016) show that upon making an investment decision in an early stage firm: 17% of venture capitalists do not use any financial metric, 23% do not adjust for risk, 31% do not forecast cash flows, 48% admit to making 'gut' investment decision. Overall, venture capitalists appear to make decisions inconsistent with financial theory.

However, by agreeing to invest, investors, whether implicitly or explicitly, assess the risk of an early-stage firm. In this paper we shows how to translate an investment to its respective implicit probability of growth and thus draw conclusions about its riskiness (priced in by investors) relative to other investments. The implicit probability of growth is a forward looking quantity that measures the relative risk upon making an investment. Our approach to measure risk is very different from traditional techniques and it is the key focus of this work; it allows us to study the risk in individual investments.

To achieve our goal, we develop a dynamic contracting framework that incorporates key determinants of an early stage firm. In the existing continuous-time agency models, typically, agent's hidden actions contemporaneously affect firm's outcome. In this paper we break the contemporaneous effect and let agent's actions affect firm's outcome in the future. This framework is particularly suitable for early-stage firms, where agent's actions today affect firm's growth in the future. In this setup, the entrepreneur faces a tradeoff between implementing two types of technologies: an existing technology that performs well in the near-term but has little future growth, and a new high-growth technology that performs poorly in the near-term but has a significant future growth. In realizations where growth takes time to materialize, long-term investors are concerned that high-growth may never materialize because the entrepreneur implemented an existing technology with little or no future growth.

As Hall and Lerner (2010) illustrate, an optimal innovation strategy has an option-like character. Even if managerial decisions are implemented immediately their outcome on firm's performance may take time to materialize. However, the literature on dynamic contracting under moral hazard assumes a contemporaneous relationship between hidden actions and firm's outcome: exerting effort or diverting cash-flow instantaneously affects firm's performance. Mitigating this contemporaneous relationship has several important applications, among them is implementing a new technology.

Recent work by Manso (2011) suggests ways to incentivize managers to experiment with new technologies. Though, in his setup experimentation has a deterministic time-horizon: there is no uncertainty about the amount of time experimentation takes. In contrast, in our setup experimentation time-horizon is random and the tension between implementing an existing and a new technology persists. Our dynamic agency problem builds upon DeMarzo and Sannikov (2006), and Biais, Mariotti, Plantin, and Rochet (2007) with modifications to accommodate key determinants of early-stage firms. Instead of hiring the manager to run a project, in our model, the entrepreneur owns a project but does not have the required capital to implement a technology. Investors are willing to contribute the required capital in exchange for owning a fraction of the firm, and the parties sign a contract that specifies contingencies for additional payments to the entrepreneur.

Our novel approach allows to map investments to their respective implicit probabilities of growth. An investment with higher implicit probability of growth is one that investors priced in that growth is more likely to materialize, which translates to a less risky investment compared to other investments. This creates a medium for comparing the risk associated with early-stage investments. For instance, recent evidence by Hellmann and Thiele (2015) suggest that the rise of the angel-investing market coincides with a shift of venture capitalists to later stage investments. One of the first questions that comes to mind is does this dynamics reduce or increase the risk bore by venture capitalists? By using the implicit probability of growth we can investigate whether an investment in a later stage is de facto less or more risky. Clearly, the time-effect reduces the risk of a later stage investment; however, competition among VCs, equity share and the amount invested may also change. The overall effect is not clear and may in fact increase risk. Investigating these tradeoffs help us to understand better the dynamics of the venture capitalists market.

Furthermore, Nanda and Rhodes-Kropf (2013) find that in hot markets VCs invest in riskier and more innovative start-ups. One can think of two potential reasons for this phenomenon: Do VCs are *intentionally* looking for riskier more innovative start-ups? Alternatively, does the flooding of cheap money create more competition among VCs that force them, *unintentionally*, to pursue riskier investments? Our methodology can help disentangle these two possibilities. If VCs are intentionally looking for higher risk profile the implicit probability of growth of their investments should decrease. However, if competition forces VCs out of their desired investments then the implicit probability of growth should not decrease or even increase. The main reason is that demand pressure pushes up prices of early-stage firms, and this translates to higher implicit probability of success.

Patient investors with a long investment horizon and a well diversified portfolio would like the entrepreneur to implement a new technology with high-growth. However, the entrepreneur is less patient, with shorter horizon and motivating her to implement a new technology requires sufficient incentives. This horizon misalignment is in line with the empirical findings in Puri and Zarutskie (2012), which claims that venture capital is patient money and venture capitalists recognize the option value in their investments and exert effort to ensure companies do not close down. In general, the horizon misalignment between investors and managers is widely accepted as a key agency friction, which goes back to Narayanan (1985), Kaplan (1988) and Stein (1989) with studies of manager's myopic behavior.

In practice, once a suitable project is found, a short document with all the pertinent information, coined term-sheet, is signed. Among other things, term sheets are comprised of standard mechanisms to achieve investors' goals. In particular aligning incentives to pursue high-growth. One standard mechanism that motivates the entrepreneur to pursue high growth is by issuing preferred stocks to investors as opposed to common stocks to the entrepreneur. Preferred stocks have a higher priority, which means investors receive their initial investment plus an additional predetermined return before any allocation is made to common stock holders — the entrepreneur. An even more powerful mechanism being employed regularly is commonly referred as 'double deep'. Intuitively, double deep means that investors' preferred stocks participate also as common stocks: after getting paid as preferred stock holders, investors participate again as common stock holders. Though, intended as risk reduction techniques for investors, in essence, these mechanisms and others like them provide a lower bound on potential returns: any project that does not have the potential to deliver returns higher than the lower bound is not worth pursing. In turn, it forces entrepreneurs to pursue projects with high growth.

Implementing a new technology requires sufficient incentives. This mandates that undesired results be met with penalties. However, due to limited liability, negative wealth is precluded. Thus, penalties accumulate until inefficient termination is triggered. There are benefits associated with early payments: as investors own a fraction of the firm, any payment retained within the firm is only worth a fraction to the entrepreneur. Furthermore, the entrepreneur's time-discount parameter, or impatience parameter, is weakly bigger than those of investors. These two sources of impatience together imply that payments to the entrepreneur are made as early as possible. In addition, there is a cost associated with early payments: it may induce more histories that trigger early termination. The optimal contract determines the boundary at which the cost is equal to the benefit.

Implementing an existing technology generates higher expected cash-flow in the near term. In a bid to avoid deviations to the existing technology, investors make these deviations non beneficial to the entrepreneur. They achieve their goal by rewarding the entrepreneur after histories of poor performance. This is in stark contrast to the standard pay-for-performance, which punishes managers for histories of poor performance by terminating their contract. The implementation phase ends randomly when the new technology reaches maturity. From this point the new technology generates the highest expected cash-flow, and incentives of investors and the entrepreneur are aligned.

This result is consistent with Manso (2011). Though, the economic mechanism that generates his result is quite different: in a static setup the entrepreneur can learn about the probability of success of new untested technologies. If success is seen in the first period, then the probability of success in the second period increases, which overall increases the expected payoff. However, as Hall and Lerner (2010) show, investment in R&D in general and in start-ups in particular has an option-like character and should not be analyzed in a static framework. At the start of a venture's life, failure has no predictive power about future success, while 'closer to maturity' it may have some indication. Exactly how much is 'closer-to maturity' cannot be identified and should be treated as a random time. Indeed, the degree of dispersion in times between inception to IPO-exits is substantial and state dependent, as Cochrane (2005), among others, illustrates. In contrast, we address this empirical fact: in a dynamic setup the implementation of a new technology takes time to materialize, is random and cannot be predicted.

The entrepreneur has a claim on firm's cash-flow. This means that exactly as a current dividend payment decreases the value of a stock, a positive cash-flow shock decreases the value of wealth. After a long enough sequence of positive cash-flow shocks, the value of wealth reaches zero and due to limited liability triggers an inefficient termination. With the same logic, a negative cash-flow shock increases the value of wealth. After a long enough sequence of negative cash-flow shocks, the value of wealth reaches a level that induces the entrepreneur to switch to the existing technology; by doing so, the entrepreneur increases the expected cash-flow and decreases the value of wealth. In anticipation for such an outcome, investors pay a cash bonus exactly at these instances, which immediately reduce the value of wealth, and thus alleviate the entrepreneur's incentives to deviate. In other words, a sequence of long enough positive cash-flow shocks excessively front-loads wealth. This induces the value of wealth to reach zero and to trigger inefficient termination. A sequence of long enough negative cash-flow shocks excessively back-load wealth. In a counteract, the entrepreneur reduces the excessive back-load by switching to an existing technology that produces higher expected cash-flow. Investors anticipate this behavior and pay a cash bonus exactly then, thus reducing the excessive back-load and eliminating the entrepreneur's incentives to deviate.

In the main setup investors are sophisticated; as such, they are able to monitor the technological progress without relying on the entrepreneur for providing information; when the high-growth technology matures, they observe it immediately and shift the entrepreneur's incentives to accommodate that growth. In Section 3.1 we mitigate that assumption and consider non-sophisticated investors; as such, they rely on the entrepreneur for providing information. The entrepreneur may choose to truthfully disclose growth arrival time or opportunistically report growth arrival time prior to or after its true arrival.

By falsely reporting growth arrival time before its true arrival, the entrepreneur avoids inefficient termination and loses the high-growth potential; it substantially reduces future returns in exchange for eliminating the risk of early termination. This strategy would be optimal when the risk of early termination is imminent; when wealth approaches the outside option. However, investors can avoid the risk of false reporting: if entrepreneur's outside option is strictly positive, investors can endogenously choose their equity share so that the threshold for false reporting is below the outside option and the entrepreneur always truthfully reports growth arrival time. Furthermore, by falsely reporting growth arrival time after its true arrival, the entrepreneur gains additional bonus payments that would otherwise not be paid; in exchange, the gains from high-growth are postponed. However, if the growth is sufficiently high the threshold for false reporting would be higher than the maximal possible wealth, and the entrepreneur would always prefer to truthfully disclose growth.

In another useful extension, growth arrives incrementally over time instead of all at once. In this case, every once in a while a milestone is reached. At these times, the firm's expected cash-flow increases; however, these increments are not enough to topple the expected cash-flow generated by the existing technology. This means that up until the last milestone is reached the entrepreneur is rewarded for poor-performance. It is only after the last milestone that the high-growth technology generates higher expected cash-flow than the existing technology and the entrepreneur's incentives change. In general, the dynamic structure of the model is flexible enough to accommodate various types of empirically relevant extensions to the main setup.

The contract is implemented by opening a limited liability firm with an initial balance equals to the initial investment, where both the entrepreneur and investors are equity holders. The firm's balance net rental cost of capital equals firm's cash-flow that is deposited into it minus cash payments that are withdrawn from it. We show that if the initial investment is above a certain threshold then firm's balance never reaches zero and it is not necessary to keep track of it. This implies that the entrepreneur's continuation payoff is sufficient to summarize the state space, thus reducing the complexity of the contract. Furthermore, it implies that the total amount investors can potentially lose, if inefficient termination is triggered, is capped and can never exceed the initial balance. In contrast, DeMarzo and Sannikov (2006) assume that investors have deep pockets, which implies that the total amount investors can potentially lose is unbounded; for some histories the total amount paid to the manager is higher than the amount invested.

First and foremost, out paper contributes to the ongoing literature on risk and return of venture capital investments. By and large, measurement of risk is calculated by looking at returns of investments to exit; either ipo, acquisition or default. However, due to scarcity of data and the econometric hurdles of selection bias there are very few papers analyzing the risk and return of private equity funds in general and venture capitalist funds in particular. The selection bias was first acknowledged by Cochrane (2005) and Hwang, Quigley, and Woodward (2005); Cochrane (2005) emphasizes the selection bias and its difficulties and correct for selection bias using a maximum likelihood estimation of returns. A more recent work by Korteweg and Sorensen (2010) extend a standard dynamic asset pricing model by adding a selection process. However, they emphasize that their results are substantially different from those obtained in Cochrane (2005). We contribute to this literature by introducing a new way to measure risk. Our methodology characterizes each individual investment's risk relative to other investments measured by their respective implicit probability of growth. It relies on the technology characteristics and correct for the incentives alignment. By measuring risk this way we eliminate any type of selection bias and hopefully allow for further investigation of venture capitalists risk and return.

Furthermore, our paper contributes to the literature on dynamic contracting. The studies that are closely related to ours are DeMarzo and Sannikov (2006) and Biais et al. (2007). Our contribution relative to their models is the disentanglement of manager's actions and firm's outcome. In our model, manager's desired actions have negative impact on firm's performance in the short-run but rather positive impact in the long-run. This characterization is particularly helpful in studying early stage firms. In such a firm, an entrepreneur is incentivized to implement a technology that has the highest future growth even at the expense of poor short-run performance. We further extend previous dynamic models by mitigating the deep-pocket assumption and allowing investors to cap the maximum amount lost without introducing additional complexity to the model.

Previous studies of incentives for innovation in a contracting framework were done in static models. We are the first to analyze the incentives for innovation in a dynamic contracting framework. By doing so we incorporate key facts about innovation in early stage firms that cannot be addressed otherwise. First and foremost, the amount of time it takes for innovation to materialize is not known and should be treated as a random variable. Second, learning about the probability a new technology succeeds is not linear: at the start of an early stage firm, failure does not have predicting power on the probability of future success. Indeed, Holmstrom (1989), illustrates that measuring innovation is hard; thus, it is very costly to pursue alongside easy-to-measure, routine activities. Aghion and Tirole (1994) extend this view by illustrating that even upon successful innovation, it is hard to predict the magnitude of success, ex-ante. We incorporate these views in our setup: the time to successful innovation and its magnitude are not known and cannot be learned.

Over the years a different approach to study innovation was developed. This approach studies innovation using Bayesian analysis. Broadly speaking, these models assume that it is productive to learn about the process of innovation. In a recent work, Manso (2011) incorporates a Bandit problem into a principal-agent framework and studies the incentives for exploration and exploitation. We argue that it may be productive to learn about innovation processes in different setups, but rather unsuitable for early-stage firms and entrepreneurs implementing innovative technologies. Another approach to study innovation was developed by Holmstrom and Milgrom (1991). In their model the agent allocates effort among different tasks. They conclude that common activities should be severely restricted for agents with innovative activities because these activities are hard to measure, which inappropriately incentivizes the agent to pursue the common activities.

The agency friction generating our results falls within a vast literature on manager's myopic behavior. Due to different reasons, managers choose to concentrate on short-term projects that may be harmful in the long-term; for example, among others, due to career concerns studied by Narayanan (1985), due to takeover threats studied by Stein (1988), due to concerns about near-term stock prices studied by Stein (1989). And in the same vein, if the entrepreneur is not properly incentivized she diverts her attention to implementing a lower growth technology that pays higher expected cash-flow in the short-term but has no future long-term growth.

The rest of this paper is organized as follows. Section 2 introduces the economic setup; Section 3 derives the optimal contract; Section 4 discusses the implementation; Section 5 introduces the implicit probability of default and its empirical predictions; Section 6 introduces two generalizations to the main economic model. In the first we increase the set of technologies and in the second we let growth arrive incrementally. In Section 7 we conclude.

2 The Economic Setup

The entrepreneur owns a startup and needs external capital $K_0 \geq 0$ to implement a technology. Investors offer to contribute this capital in exchange for a fraction $(1-q)$ of the firm and a contract specified below. For now, let us assume that there are only two technologies: a new high-growth technology $\bar{k}$, and an existing no-growth technology $\underline{k}$. The new high-growth technology produces cash-flow characterized by

$$dX_{\bar{k}t} = \left(\mu_{\bar{k}} + \alpha_{\bar{k}} \mathbf{1}_{\tau_{\bar{k}}}\right) dt + \sigma dZ_t, \tag{1}$$

and the existing no-growth technology produces cash-flow characterized by

$$dX_{\underline{k}t} = \mu_{\underline{k}} dt + \sigma dZ_t, \tag{2}$$

where $\mu_{\bar{k}}, \alpha_{\bar{k}}, \mu_{\underline{k}}, \sigma > 0$ are positive constants; Z_t is a standard Brownian motion; $\tau_{\bar{k}}$ is a stopping time that captures the *growth arrival time* [1]; the parameter $\alpha_{\bar{k}}$ captures the *growth's magnitude*; and $\mathbf{1}_{\tau_{\bar{k}}}$ is characterized by

$$\mathbf{1}_{\tau_{\bar{k}}} = \begin{cases} 0 & \text{if } t < \tau_{\bar{k}} \\ 1 & \text{if } t \geq \tau_{\bar{k}} \end{cases}, \tag{3}$$

and captures the information in $\tau_{\bar{k}}$; we assume that $\tau_{\bar{k}} < \infty$ probability almost surely.

Entrepreneur's equity share q as well as the initial amount invested K_0 are exogenous and taken as parameters for the model. Of course, for a fixed valuation, there is a tradeoff between these two quantities: higher initial amount invested implies lower equity share remained for the entrepreneur. These parameters can be the result of negotiation power and market conditions. In Section 3.1 we add an additional structure to the model and endogenize the equity share.

Agency Friction:

By switching from the no-growth technology to the high-growth technology the entrepreneur forgoes immediate expected cash-flow,

$$\mu_{\underline{k}} - \mu_{\bar{k}} > 0,$$

in exchange for a risky gamble on high-growth in the future. If growth materializes, expected cash-flow increases substantially,

$$\mu_{\bar{k}} + \alpha_{\bar{k}} - \mu_{\underline{k}} > 0,$$

[1] The stopping time is with respect to the filtration generated by Z_t; though, this is not important for the economic setup. Furthermore, both parties know exactly when $\tau_{\bar{k}}$ occurs but it does not reveal information about Z_t if the rule governing $\tau_{\bar{k}}$ is not known.

but the chance of that happening is slim. Investors prefer a risky, high growth investment profile and therefore, they invest in the entrepreneur's project. However, the entrepreneur prefers early payments and motivating her to postpone payments and pursue high-growth requires sufficient incentives.

Entrepreneur's preference towards early payments arises from two distinct sources. First, through a direct effect: the entrepreneur's wealth is non-negative and her impatient parameter is higher, $\gamma \geq r$. Second, through an indirect effect: investors' equity ownership implies that every unit of capital that is not paid in salary is worth only a fraction q to the entrepreneur if retain within the firm. These two sources imply that cash transfers are made as early as possible and, more importantly, they generate incentives misalignment; it implies that the entrepreneur has incentives to implement an existing technology with higher immediate expected return [2].

Information:
Investors observe two signals: the cash-flow reported by the entrepreneur

$$dY_t = dX_{\bar{k}t} \quad \text{or} \quad dY_t = dX_{\underline{k}t}, \quad \forall t \in [0, T]. \tag{4}$$

and the growth arrival time $1_{\tau_{\bar{k}}}$. When a technology materializes it becomes a public knowledge and everyone learns about its materialization. This assumption is reasonable for many different empirical cases because materialization of technologies is usually of public interest. For example, news about the progress of: cancer treatments, self-driving cars technology, storing energy, computational power technology, reaches the public upon materialization. In Section 3.1 we alleviate this assumption and let the entrepreneur self-report when the technology matures; it creates additional source of complexity that we postpone for a later discussion.

The entrepreneur decides on which technology to implement; this decision is private and is not observed by investors; similar to investors, the entrepreneur observes $1_{\tau_{\bar{k}}}$.

Contract:
The entrepreneur and investors sign a contract (I_t, T), that specifies a termination policy T that may be random and payments $\{I_t,\ 0 \leq t \leq T\}$ that are based on the reported cash-flow, Y_t and the state of innovation, $1_{\tau_{\bar{k}}}$; the entrepreneur's total expected payoff is given by

$$W_0 = E\left[\int_0^T e^{-\gamma s}\Big\{ \underbrace{q\Big(dY_s - dI_s\Big)}_{\text{claim on firm's cash-flow}} + \underbrace{dI_s}_{\text{cash payments}} \Big\} + e^{-rT}R\right], \tag{5}$$

[2] Different from DeMarzo and Sannikov (2006), weakly impatience, $\gamma \geq r$, also leads to the same optimal contract because postponing payments generate a cost for the entrepreneur through the indirect effect.

and the investors' total expected payoff is given by

$$V(W) = E\left[\int_0^T e^{-rs}\Big\{\underbrace{(1-q)\Big(dY_s - dI_s\Big)}_{\text{claim on firm's cash-flow}}\Big\} + e^{-rT}L \mid W = W_0\right]. \tag{6}$$

From investors' perspective, they offer a contract (I,T) that specifies a termination policy and payments $\{I_t;\ 0 \leq t \leq T\}$ based on reported cash-flow Y. From the entrepreneur's perspective, for a given contract (I,T) she chooses a technology to maximize her payoff. The implemented technology is incentive compatible if it maximizes her total expected payoff W_0 for a given contract (I,T). An incentive compatible contract is the quadruple $(I,T,X_{\bar{k}t})$ that includes the entrepreneur recommended technology. We assume that it is optimal to implement high-growth technology all the time, and verify its optimality in Section 3. Therefore, unless otherwise is specified, the expectation operator is taken under the measure induced by $dX_{\bar{k}t}$.

3 Optimal Contract

The high-growth technology has two regimes: before and after high-growth materializes. High-growth arrival time determines in which regime we are currently at, which is denoted by the indicator function $\mathbf{1}_{\tau_{\bar{k}}}$. Furthermore, both the entrepreneur and investors learn *independently* about that time by observing this indicator function. This means that the entrepreneur cannot manipulate investors about growth arrival time and the incentives problem can be decomposed to two separate problems: the implementation phase and post-implementation phase problems. Later in the chapter we discuss the case where only the entrepreneur observes the indicator function, and investors rely on the entrepreneur for obtaining that information.

Incentive Compatibility:
Let $W_t\,(Y,\tau_{\bar{k}})$ denotes the promised value that the entrepreneur receives from transfers after a history of reports $(Y_t,\ 0 \leq t \leq T)$ and growth realization $\mathbf{1}_{\tau_{\bar{k}}}$, such that she implements high-growth from time t onward.

$$W_t\,(Y,\tau_{\bar{k}}) = E_t\left[\int_t^T e^{-\gamma(s-t)}\Big\{q dY_s + (1-q)\,dI_s\Big\} + e^{-rT}R\right]. \tag{7}$$

Thanks to the Martingale Representation Theorem we can rewrite the continuation value W_t, Equation (7), in terms of the observable performance dY_t, Equation (8), summarized in the following Lemma.

Lemma 1. *If the entrepreneur follows the recommended high-growth technology, the entrepreneur's* *continuation payoff can be represented as*

$$dW_t = \begin{cases} \gamma W_t dt - (1-q)\, dI_t - q\mu_{\bar{k}} dt + (\beta_t - q)\left(dY_t - \mu_{\bar{k}} dt\right) & \textit{if} \;\; t < \tau_{\bar{k}} \\ \gamma W_t dt - (1-q)\, dI_t - q\left(\mu_{\bar{k}} + \alpha_{\bar{k}}\right) dt + (\beta_t - q)\left(dY_t - \left(\mu_{\bar{k}} + \alpha_{\bar{k}}\right) dt\right) & \textit{if} \;\; t \geq \tau_{\bar{k}} \end{cases} \tag{8}$$

and β_t *is a function of the history of reports,* $(Y_s, 0 \leq s \leq t)$*, and growth arrival time,* $\tau_{\bar{k}}$*.*

Based on the observable reports investors make sure that the entrepreneur is not better off implementing the low-growth technology. They achieve their goal by controlling the entrepreneur's exposure to cash-flow shocks ,which is done by determining the appropriate β_t function such that switching to any feasible technology has negative expected impact on wealth. Lemma 2 implies that implementing the high-growth technology, $\bar{k}$, is incentive compatible for a range of values of β_t, and that range depends on whether growth has materialized.

Lemma 2. *The high-growth technology* $\bar{k}$ *is incentive compatible if and only if both conditions*

1. $\beta_t \leq 0, \quad \textit{if} \;\; t < \tau_{\bar{k}}$
2. $\beta_t \geq 0, \quad \textit{if} \;\; t \geq \tau_{\bar{k}}$

are satisfied for all $t \leq T$*.*

An analogy to asset pricing clarifies the economics behind the incentive compatibility condition. A stock is a claim on dividend; as such, when a dividend payment is made stock price decreases immediately. With the same logic, the entrepreneur has a claim on the firm's cash-flow; as such, when a positive cash-flow shock occurs, the entrepreneur's wealth decreases immediately. Investors can offset or increase this exposure by the control β_t. If $\beta_t \leq 0$, investors would further decrease the entrepreneur's initial exposure to positive cash-flow shocks, while if $\beta_t \geq 0$, investors would like to offset the entrepreneur's initial exposure.

During the implementation phase, before growth materializes, investors decrease the entrepreneur's exposure to shocks by $\beta_t \leq 0$. During that period negative cash-flow shocks back-load the entrepreneur's wealth. If the entrepreneur deviates from $\bar{k}$ to $\underline{k}$, she loses $\beta_t \mu_{\underline{k}}$ and gains $-\beta_t \mu_{\bar{k}}$. During the post-implementation phase, after growth materializes, investors would like to offset entrepreneur's initial negative exposure by $\beta_t \geq 0$. If the entrepreneur deviates from $\bar{k}$ to $\underline{k}$, she gains $\beta_t \mu_{\underline{k}}$, and loses $-\beta_t\left(\mu_{\bar{k}} + \alpha_{\bar{k}}\right)$. Figure 1 illustrates the tradeoff between the low- and high-growth technologies.

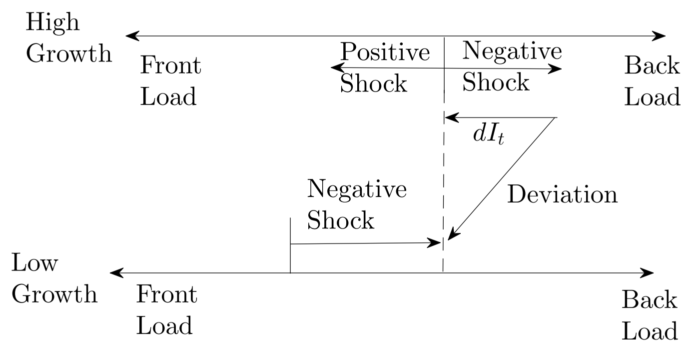


Figure 1. Because the entrepreneur has a claim on firm's cash-flow, it is illustrative to think about the entrepreneur's wealth in terms of dividend payout and a stock price. The low-growth technology has higher expected payoff implying that cash-flow generated by this technology front-loads entrepreneur's wealth more than high-growth does. A bad shock decreases the current cash-flow implying that the cash-flow is back-loading. The entrepreneur is impatient and if back loading is severe she deviates to the low-growth technology to reduce the back-loading. Investors anticipate that and pay cash bonuses exactly when back-loading is severe. The optimal contract determines that threshold.

Patience $\gamma \geq r$:
The mere fact that the entrepreneur has an equity stake in the firm implies that every unit of capital that is not transferred immediately would worth only a fraction q if growth materializes and incentives change. Therefore, even when investors and the entrepreneur are equally patient, there is an uncertainty cost bore by the entrepreneur. It is the cost associated with the shift in incentives if growth materializes. In this case, the entrepreneur obtain only a fraction q of the retained transfer, dI_t. Thus, it is never optimal to postpone payments indefinitely. This result is in contrast to DeMarzo and Sannikov (2006). Their setting requires that the agent is strictly less patient than the principal, $\gamma > r$ and when $\gamma = r$ the promise keeping condition is violated: by pushing dI_t a bit further, the principal benefits from decreasing the probability of early termination without any cost bore by the manager. It is equally costly to the manager to have the principal save the payments in het account, as long as the promise keeping condition is not violated. This implies that the principal would like to postpone the payment dI_t indefinitely, rendering $dI_t = 0$ and $W_t = 0$ always, implying that the optimal contract does not exist. In addition, He (2009) shows that when firm's cash-flow follow a geometric Brownian motion, the optimal contract exists also when $\gamma = r$. In his model, because agent's continuation payoff is always positive there is no future liquidation and the first-best is achieved.

Investors:
We denote by $V_i(W_t)$ the value function of investors; the highest profits that can be attained for any given payoff W_t to the entrepreneur, contingent on the progress of the technology, $i \in \{1, 2\}$. Before growth materializes, $i = 1$. When the entrepreneur implements the recommended highest growth technology investors set $\beta_t = 0$ and the value function takes the form

$$rV_1 = (1-q)\,\mu_{\bar{k}} + (\gamma W - q\mu_{\bar{k}})\,V_1' + \frac{1}{2}\sigma^2 q^2 V_1''. \tag{9}$$

Cash payments are transferred from the firm to the entrepreneur when W_t crosses W_*. The size of the payment is such that brings W_t back to W_*. Paying the entrepreneur cash to reduce the continuation payoff implies two boundary conditions: the smooth pasting and the super-contact conditions:

$$V_1'\,(W_*) = -1, \tag{10}$$

$$V_1''\,(W_*) = 0. \tag{11}$$

Applying these conditions to Equation (9) implies that at W_*

$$rV_1 + \gamma W = \mu_{\bar{k}}.$$

Furthermore, if W_t reaches the outside option R the firm is liquidated, and investors attain L,

$$V_1(R) = L. \tag{12}$$

Once growth materializes, $i = 2$ and investors shift incentives by setting $\beta_t = q$. From this point forward there is no agency friction and the first best contract is achieved.

$$V_2\,(W) = v_2 \equiv (1-q)\,\frac{\mu_{\bar{k}} + \alpha_{\bar{k}}}{r}, \qquad W_t = w_2 \equiv q\frac{\mu_{\bar{k}} + \alpha_{\bar{k}}}{\gamma}. \tag{13}$$

The value to investors and the entrepreneur are their corresponding equity share of the discounted expected cash-flow. These quantities are state independent and remain v_2, w_2 indefinitely, and $dI_t = 0$. The following proposition summarizes these results.

Proposition 1. *During the implementation phase ($t < \tau_{\bar{k}}$) W_t evolves according to*

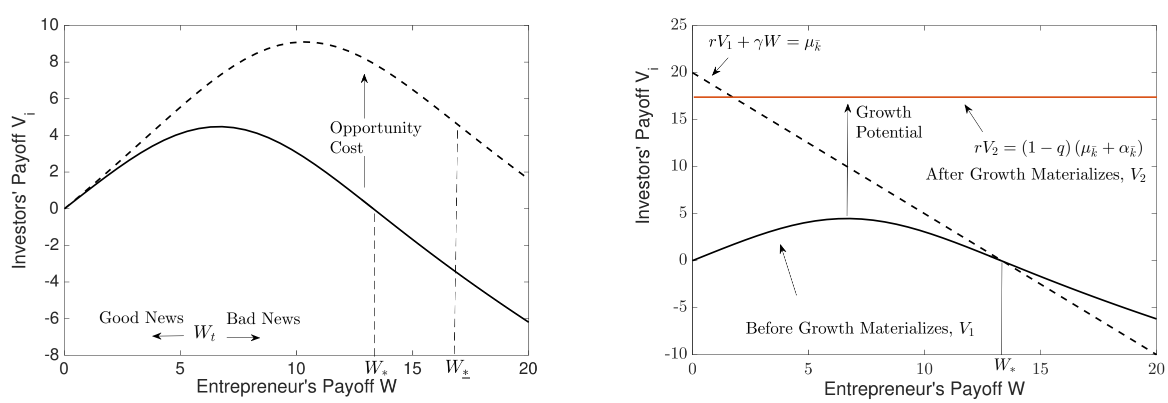
$$dW_t = \gamma W_t dt - q\mu_{\bar{k}} dt - q\,(dY_t - \mu_{\bar{k}} dt) \tag{14}$$

when $W_t \in [R, W_*]$ and $dI_t = 0$. When $W_t > W_*$ a cash payments dI_t is made so that $W_t = W_*$. Investors expected payoff is concave and follows the ordinary differential equation given in Equation (9) subject to the smooth pasting, super-contract and initial conditions given in Equations (10), (11) and (12); the contract is terminated when W_t reaches R denoted by time T. After the implementa-

tion phase ($t \geq \tau_{\bar{k}}$) investors expected payoff reaches the first-best absorbing state given in Equation (13).

The optimal contract suggests that investors transfer cash bonuses, dI_t, after histories of bad performance. Because a shift to the no-growth technology immediately increases firm's cash-flow, it is after a history of bad performance that no-growth technology becomes attractive, as illustrated in Figure 1. To lessen its attractiveness, investors transfer cash bonuses exactly in these instances. Thus, knowing that cash bonuses arrive after histories of bad performance, the entrepreneur is better off continuing to implement the high-growth technology through bad times. Because the entrepreneur owns an equity share of the firm, good performance front-load wealth. Thus, it is after good performance that her equity share may front-load so much that wealth falls below the outside option, causing the contract to be terminated.

In Figure 2 we illustrate an example of the optimal contract. In the left panel, Figure 2a, we illustrate the implicit cost of implementing high-growth. It is rather a form of opportunity cost: it is the cost associated with implementing high-growth relative to all other alternatives, which in this case is the low-growth technology. This cost measures the loss in value for investors during the implementation phase in exchange for having the opportunity of high future growth. Indeed, the only reason to forgo the higher expected cash-flow of the no-growth technology is the opportunity for much higher expected cash-flow in the future if growth materializes. In other words, without the growth opportunity investors are better off implementing low-growth. In the right panel, Figure 2b, we illustrate the jump in value to investors if growth materializes'. Furthermore, the point W_* is the optimal cash payment threshold. It is the threshold that determines how poor the history of cash-flow should be so that the entrepreneur is compensated no to deviate to the no-growth technology.



(a) (b)

Figure 2. The optimal contract. For $L = R = 0$, $\mu_{\bar{k}} = 2$, $\alpha_{\bar{k}} = 1.5$, $q = 50\%$, $\mu_{\underline{k}} = 3$, $\alpha_{\underline{k}} = 0$, $r = 10\%$, $\gamma = 15\%$, and $\sigma = 4$. $W_{\underline{*}}$ is the reflection point associated with technology $\underline{k}$.

Figure 2a also shows that the employment region is smaller for high growth technology, $[R, W_*] \subset [\,R, W_{\underline{*}}\,]$. [3] Intuitively, when the expected cash-flow is low, their discounted value decreases and so is investors' value function. However, this should *not* to be interpreted as higher probability of default or higher frequency of payments. There are two opposing forces at play: when the entrepreneur implements the no-growth technology, firm's expected cash-flow increase, which front-loads wealth. Front loading wealth implies that there are higher chances of hitting the outside option, but at the same time, the employment region increases reducing the chances of hitting the outside option. It is not entirely clear which force has a higher impact.

Moreover, it is essential that the entrepreneur owns a fraction of the firm, $q > 0$. If investors buy 100% of the firm and let the entrepreneur to run it, as is the case in the traditional setup and also in DeMarzo and Sannikov (2006), it would be impossible to provide sufficient incentives for the entrepreneur to implement high-growth technology, as the only upside from forfeiting the high expected cash-flow of no-growth technology is the opportunity of having a fraction of all the future cash-flow, if growth materializes. This upside can be captured only if $q > 0$. In other words, under the optimal contract, if $\beta_t = q = 0$, investors are unable to expose the entrepreneur to cash-flow shocks, thus would not be able to provide sufficient incentives to implement the high growth technology. The following Proposition summarizes this finding.

Proposition 2. *The optimal contract does not exist if the entrepreneur does not have an equity share, $q = 0$.*

3.1 Truthful Reporting of Growth Arrival

In the main economic setup, both investors and the entrepreneur receive a signal when the high-growth technology materializes, $\mathbf{1}_{\tau_{\bar{k}}}$. In this section, we assume that *only* the entrepreneur receives that signal. In order to adjust incentives, investors rely on the entrepreneur to truthfully disclose that information. However, the entrepreneur can choose between three potential disclosure options: a truthful on-time, a premature and an overdue disclosure options. When the entrepreneur announces that the technology matures, investors change incentives, the agency friction is eliminated and the first-best absorbing state is achieved. At this time, the parties split the expected discounted future cash-flow relative to their equity share.

First, let us investigate the tradeoff between on-time and overdue disclosures; by overdue disclose, the entrepreneur remains exposed to the risk of early termination, but if the growth potential is only marginal, current wealth may be higher. More formally, the inequality

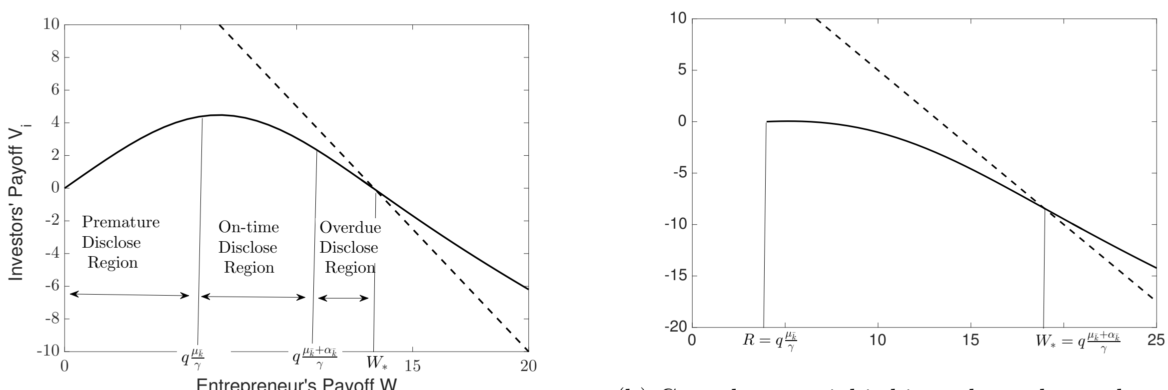
$$\underbrace{q\frac{\mu_{\bar{k}} + \alpha_{\bar{k}}}{\gamma}}_{\text{on-time disclose}} \geq \underbrace{W_t}_{\text{overdue disclose}} \tag{15}$$

[3] This comparative statics result is borrowed from in DeMarzo and Sannikov (2006). They show that W_* decreases as the drift of firm's cash-flow decreases; thus $\mu_{\bar{k}} < \mu_{\underline{k}}$ implies that $W_* < W_{\underline{*}}$ and $[R, W_*] \subset [R, W_{\underline{*}}]$.

has to be satisfied for all histories, probability almost surely. Careful investigation shows that for any given share q, the highest possible wealth, W_*, satisfies

$$W_* \geq q\frac{\mu_{\bar{k}}}{\gamma}. \tag{16}$$

This condition implies that if the growth potential is small, for any equity share, overdue disclosure may occur for some histories. Thus, the restriction in Inequality (15) is a restriction on the minimal growth potential to induce truthful reporting. Intuitively, the total surplus from pursuing high-growth is obtained by the line $\mu_{\bar{k}} = rV_1(W) + \gamma W$. Each point on this line defines a split of the total surplus between the two parties; the reflection point, W_*, as a point on the total surplus line also defines a split. Without agency friction each party consume its corresponding equity share; however, due to agency friction, investors have to transfer funds to the entrepreneur to align incentives. This implies that out of the total surplus, at W_*, the entrepreneur total wealth is higher than her equity share, justifying Inequality (16). If the growth potential, $\alpha_{\bar{k}}$ is too small, Inequality (15) is not satisfied and overdue disclosure may occur in states where entrepreneur's wealth runs high. This result is illustrated in Figure 3.



(a) Growth potential is small and overdue disclosure may occur, $\mu_{\bar{k}} = 2$, $\alpha_{\bar{k}} = 1.5$, $R = 0$.

(b) Growth potential is big and overdue and premature disclosures never occur, $\mu_{\bar{k}} = 1.2$, $\alpha_{\bar{k}} = 4.5$, $R = 4$, $W_* \approx 19$.

Figure 3. The optimal contract. For $L = 0$, $q = 50\%$, $r = 10\%$, $\gamma = 15\%$, and $\sigma = 4$.

Suppose growth potential satisfies the restriction in Inequality (15), then overdue disclosure option is not optimal and the entrepreneur has to choose among premature and on-time disclosure options. The benefit of premature disclosure is in the elimination of the early-termination risk and the cost is in the loss of the future growth potential. Thus, for on-time, truthful disclosure to be optimal the inequality

$$\underbrace{q\frac{\mu_{\bar{k}}}{\gamma}}_{\text{premature disclose}} \leq \underbrace{P_t\left[\tau_{\bar{k}} < T\right]\left\{q\frac{\mu_{\bar{k}} + \alpha_{\bar{k}}}{\gamma}\right\} + \Big(1 - P_t\left[\tau_{\bar{k}} < T\right]\Big)W_t}_{\text{on-time disclose}} \tag{17}$$

has to be satisfied for any history, where $P_t\left[\tau_{\bar{k}} < T\right]$ is the probability that growth arrives before termination is triggered. If we assume independence between growth arrival ,$\tau_{\bar{k}}$, and firm's cash-flow shocks, dZ_t, then as W_t approaches the outside option, $W_t \downarrow R$ the probability of growth arrival before termination decreases $P_t\left[\tau_{\bar{k}} < T\right] \downarrow 0$. Let us define the *truth telling* threshold as $q\frac{\mu_{\bar{k}}}{\gamma}$. If wealth falls below that threshold, $W_t < q\frac{\mu_{\bar{k}}}{\gamma}$, the truth-telling condition in Inequlality (17) is not necessarily satisfied.

Intuitively, if $W_t < q\frac{\mu_{\bar{k}}}{\gamma}$, then when the probability of early termination is sufficiently high the entrepreneur is better off exercising the premature disclosure option, obtaining $q\frac{\mu_{\bar{k}}}{\gamma}$ and not risking the chance of early termination. Certainly, when the entrepreneur's wealth is above the truth telling threshold, $W_t \geq q\frac{\mu_{\bar{k}}}{\gamma}$, then the truth-telling condition in Equation (17) is always satisfied and it is optimal to truthfully report growth. Thus, investors can be sure that the entrepreneur implements the truth telling strategy if $R \geq q\frac{\mu_{\bar{k}}}{\gamma}$. This implies that for a given positive outside option, $R > 0$, the maximum equity share to induce truthful reporting can be obtained endogenously so that

$$q \leq \underbrace{R}_{\substack{\text{loss from}\\ \text{premature}\\ \text{disclose}}} \Bigg/ \underbrace{\frac{\mu_{\bar{k}}}{\gamma}}_{\substack{\text{gain from}\\ \text{premature}\\ \text{disclose}}} . \tag{18}$$

Proposition 2 claims that the equity share has to be positive, $q > 0$, to induce the entrepreneur to pursue long term growth. The Inequality (18) claims that q has to be bounded from above for truth-telling to be optimal, so that $q \leq R/\frac{\mu_{\bar{k}}}{\gamma}$. It intuitively states that the lower the outside option the lower the equity share, and the lower the value of the firm under premature disclosure the higher the equity share. The following Proposition summarizes our findings.

The restriction on the minimum growth potential, Inequality (15), and the restriction on the maximum equity share in Inequality (18) are two powerful predictions. In the first, we anticipate the minimum growth potential that is required by investment, and in the second, we predict what should be the maximum equity allocation to the entrepreneur to trigger truthful reporting.

Proposition 3. *If the outside option is strictly larger than zero, $R > 0$, then truthful reporting of growth arrival time, $\mathbf{1}_{\tau_{\bar{k}}}$, is always optimal if and only if $q\frac{\mu_{\bar{k}}+\alpha_{\bar{k}}}{\gamma} \geq W_* > R \geq q\frac{\mu_{\bar{k}}}{\gamma}$. The maximal admissible equity share is $q_* = R\frac{\gamma}{\mu_{\bar{k}}}$ and the minimum admissible growth potential satisfies $q_*\frac{\mu_{\bar{k}}+\alpha_*}{\gamma} = W_*$*

Further Discussion: Implementing High Growth in Mature Firms

Negative cash-flow shocks imply investors own a fraction of a smaller cash-flow stream. This direct effect of poor cash-flow stream is amplified in early-stage firms through an indirect effect. It is exactly after poor cash-flow stream that cash bonuses are made to the entrepreneur to align incentives. This feature may sheds some light on why matured firms are disadvantageous in developing high-growth technologies from within and why innovation usually happens in start-ups. In matured firms pay-for-performance is an optimal compensation scheme; however, this scheme leads to an adverse effect on innovative tasks, such as implementation of a high growth technology. It appears that it would be optimal to split the two tasks to different groups to avoid the conflict. In the same vein, Holmstrom (1989) argues that incorporating innovative tasks with routine tasks is very costly leading to the conclusion that matured firms are in a comparative disadvantage in conducting such tasks. Furthermore, Holmstrom and Milgrom (1991) show that in a multi-task setting, pay-for-performance in one task may draw excessive resources from other tasks, rendering it sub-optimal.

4 Implementation

The entrepreneur and investors open a limited liability firm and associate it with a bank account, K_t. The entrepreneur owns q shares and investors own the rest $1-q$ shares of the firm. Firm's cash-flow dY_t is flowing into the account and cash transfers, dI_t, are withdrawn from it, such that

$$dK_t \quad - \underbrace{\gamma K_t dt}_{\text{rental cost of capital}} = \underbrace{dY_t}_{\text{cash-flow}} - \underbrace{dI_t}_{\text{payments}}, \quad K_0 > 0, \tag{19}$$

with initial capital injection K_0 made by investors. Both the entrepreneur and investors observe the bank account, and the limited liability restriction implies that firm's bank account is non-negative at all times, $K_t \geq 0$, $\forall t < T$. [4] The rental cost of capital is $\gamma K_t dt$ and is deducted from both parties proportional to their equity share, so that the entrepreneur payment equals $q\gamma K_t dt$ and investors payments equals $(1-q)\gamma K_t dt$. It is important for tractability purposes that the rental-cost rate equals to γ. Otherwise, the bank account balance, K_t, introduces an additional state variable. We extend this discussion later in the chapter.

A priori, because there are possible histories where $K_t = 0$, K_t is an additional state variables. If such a history occurs it triggers an inefficient termination (default) because there are no more funds to be paid to the entrepreneur to align incentives. However, Proposition 4 shows that under a certain condition it is sufficient to keep track of W_t. This means that inefficient termination is triggered only by the entrepreneur's limited liability restriction, $W_t = 0$ and histories where the firm defaults do not occur unless the entrepreneur's limited liability restriction is reached,

$$K_t = 0 \quad \Longrightarrow \quad W_s = 0, \quad s \leq t. \tag{20}$$

Towards that goal, we observe that before growth materializes a positive cash-flow shock affects negatively on the entrepreneur's wealth, and positively on q shares of the bank account, such that

[4] This is in contrast to DeMarzo and Sannikov (2006) where investors have 'deep pockets', and funds are never depleted.

these two effects exactly offset each other. Let us define an auxiliary process,

$$c_t \equiv \underbrace{W_t}_{\text{Wealth}} + \underbrace{qK_t}_{\text{Share}}, \tag{21}$$

where c_t evolves according to

$$dc_t = \gamma c_t dt - dI_t + \beta_t \sigma dZ_t. \tag{22}$$

During the implementation phase $\beta_t = 0$ and c_t is not exposed to shocks; it continuously increases by $\gamma c_t dt$, and when payments are made it decreases by the amount dI_t.

Importantly, when the auxiliary process increases by $\gamma c_t dt$, the increase can be decomposed to a proportional increase in qK_t and W_t by $\gamma qK_t dt$ and $\gamma W_t dt$, respectively. Furthermore, when a payment dI_t is withdrawn from c_t; a fraction $1-q$ of the payment is associated with W_t, and a fraction q with qK_t, as shown in the evolutions of W_t in Equation (8) and K_t in Equation (19). This implies that the initial condition (W_0, qK_0) provides enough information to determine whether the entrepreneur's limited liability restriction $W_t \geq 0$ binds before the firm's limited liability restriction does. Figure 4 shows the evolution of c_t over the state-space plane, (W_t, qK_t). It illustrates that for high enough capital injection, W_t summarizes all the relevant information in K_t and it is a sufficient state variable. Proposition 4 summarizes this result.

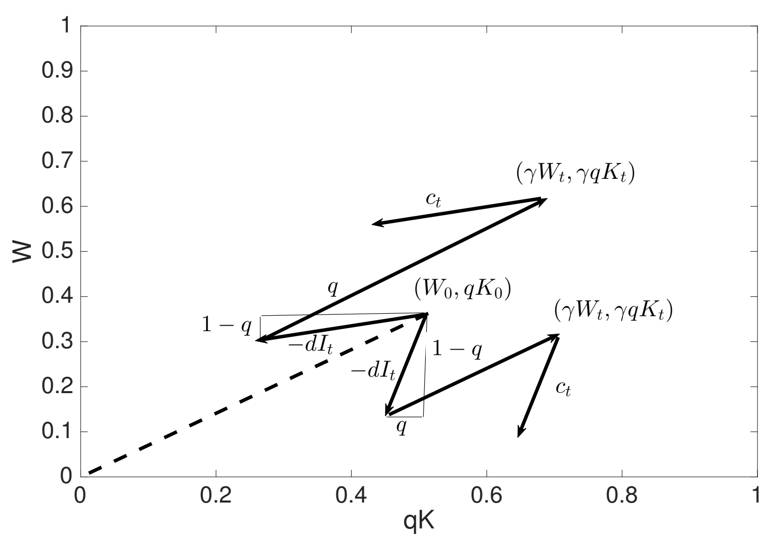


Figure 4. The evolution of c_t never crosses the line from the origin to the point (W_0, qK_0). Whether the vector c_t is above or below that line determines whether $K_t \geq 0$ or $W_t \geq 0$ restrictions binds, respectively.

Proposition 4. *If and only if $W_0 \leq (1-q)\,K_0$, then the only relevant state variable is W_t and the firm's limited liability restriction, $K_t = 0$, never binds.*

By introducing the bank account K_t, the investors commit a-priori to the maximum amount lost in the worst case scenario. At first glance, this may appear as an unnecessary restriction on the contract space. With such restriction investors are unable to contract on histories that have negative bank account balance and positive promised value. However, as Proposition 4 suggests, if the initial capital injection K_0 is sufficiently high, $\frac{W_0}{1-q} \leq K_0$, the firm's bank account is always positive, and that additional restriction never binds. [5]

5 The Implicit Probability of Growth

In this section we discuss the implicit probability of growth and its implications. The implicit probability of growth of an investment is that probability that when using it to price an investment in an early stage firm we obtain a theoretical investment identical to the one made by investors. In other words, the implicit probability of growth is the probability priced in by investors that growth materializes implicit in their investment.

The implicit probability of growth of an investment is a measurement of the risk bore by investors. By taking positions in more risky investments, the chances that growth materializes are smaller, which should be implicit in their investment. Thus, a riskier investment should have a lower implicit probability of growth. Ultimately, by measuring the risk profile of VCs with the implicit probability of growth we allow for new testable implications.

For instance, Nanda and Rhodes-Kropf (2013) find that in hot markets VCs invest in riskier and more innovative start-ups. One can think of two potential reasons for this phenomenon: Do VCs are *intentionally* looking for riskier more innovative start-ups? Alternatively, does the flooding of cheap money create more competition among VCs that force them, *unintentionally*, to pursue riskier investments? Our methodology can help disentangle these two possibilities. If VCs are intentionally looking for higher risk profile the implicit probability of growth of their investments should decrease. However, if competition forces VCs out of their desired investments then the implicit probability of growth should not decrease or even increase. The main reason is that demand pressure pushes up prices of early-stage firms, and this translates to higher implicit probability of success.

Even further, Hellmann and Thiele (2015) suggest that the rise of the angel-investing market coincides with a shift of venture capitalists to later stage investments. On the one hand, as time goes on the chances that growth materializes increase, which translates to lower risk. On the other hand, other quantities that affect the risk change as well, such as, the amount invested, the equity share and competition among VCs. By measuring risk with the implicit probability of growth we take into account all those effects. The overall effect may reduce risk but it may also increase it.

[5] The contract space is all the histories of the pair $\{(W_t, K_t), 0 \geq t \geq T\}$ such that both $K_t \geq 0$ and $W_t \geq 0$.

In the sensitivity analysis we show how changes to these different quantities change the implicit probability of growth, which translates to changes in the risk bore by investors.

5.1 Deriving The Implicit Probability of Growth

The value functions V_1 and V_2 summarizes the firm's value under the two mutually exclusive regimes: before and after growth materializes. This implies that $1-q$ shares of the firm equal a weighted average of these value functions, weighted by the probabilities of being in each of these regimes, such that

$$V(W_t) = \Big(1 - P_t\left[\tau_{\bar{k}} < T\ \right]\Big)\frac{V_1\left(W_t\right)}{1-q} + \Big(P_t\left[\tau_{\bar{k}} < T\right]\Big)\frac{V_2\left(W_t\right)}{1-q}, \quad 0 < t < \tau_{\bar{k}}, \tag{23}$$

where $P_t\left[\tau_{\bar{k}} < T\ \right]$ is the probability that growth materializes before the contract is terminated, given the information available at time t. In other words, for a given wealth W_t, $P_t\left[\tau_{\bar{k}} < T\ \right]$ summarizes all the future histories where the stopping time $\tau_{\bar{k}}$ occurs before W_t reaches R; the closer (further away) W_t to R the lower (higher) this probability. The value $V(W_t)$ in Equation (23) is the true value of the firm; it is the value obtained if both W_t and the law governing $\tau_{\bar{k}}$ are known. However, the optimal contract does not require investors nor the entrepreneur to know the law governing $\tau_{\bar{k}}$, but rather it is sufficient to observe when growth materializes, $\mathbf{1}_{\tau_{\bar{k}}}$. Without knowing the law governing $\tau_{\bar{k}}$ each party establishes an assessment of the probability of growth, which may be different from the true probability.

Let us denote investors' assessment by $\tilde{P}_t\left[\tau_{\bar{k}} < T\ \right]$; an optimistic assessment implies that $P_t\left[\tau_{\bar{k}} < T\ \right] < \tilde{P}_t\left[\tau_{\bar{k}} < T\ \right]$ and a pessimistic assessment implies that $P_t\left[\tau_{\bar{k}} < T\ \right] > \tilde{P}_t\left[\tau_{\bar{k}} < T\ \right]$. Furthermore, let us denote firm's value associated with this assessment by $\tilde{V}(W_t)$. At the onset, investors have agreed to buy $1-q$ shares of the firm in exchange for supplying the capital K_0. This investment reflects their valuation of the firm, such that

$$\underbrace{K_0}_{\text{Amount Invested}} \leq \underbrace{(1-q)\,\tilde{V}\left(W_0\right)}_{\text{Investors' Valuation}}. \tag{24}$$

For the sake of simplicity let us assume that the market for early-stage type investments is competitive and the relationship in Equation (24) holds with equality. This relation essentially links the initial investments K_0 and the implicit probability of growth. By plugging Equation (24) into Equation (23) we find that

$$\tilde{P}_0\left[\tau_{\bar{k}} < T\ \right] = \frac{K_0 - V_1\left(W_0\right)}{V_2\left(W_0\right) - V_1\left(W_0\right)}. \tag{25}$$

It is clear from Equation (25) that for a given W_0, $V_1\left(W_0\right) \leq K_0 \leq V_2\left(W_0\right)$, so that $\tilde{P}_0\left[\tau_{\bar{k}} < T\ \right]$ be a probability. If $V_1\left(W_0\right) > K_0$ or $K_0 > V_2\left(W_0\right)$ the implicit probability is essentially setup to zero or one, respectively. Furthermore, this equation shows exactly how an investment K_0 is mapped to an implicit probability of growth $\tilde{P}_0 \left[\tau_{\bar{k}} < T\ \right]$, for a given technology. Going further, we construct the relation between K_0 and W_0. The entrepreneur claims q shares of the firm, which implies that

$$W_0 = q\tilde{V}\left(W_0\right) = \frac{q}{1-q}K_0. \tag{26}$$

Accommodating Proposition 4 we require that $W_0 \leq (1-q)\,K_0$, which implies that the entrepreneur's share has to be low enough, such that

$$0 < q \leq \frac{3-\sqrt{5}}{2} \approx 38\%. \tag{27}$$

This result implies that by keeping the entrepreneur's share sufficiently small, early termination can occur only if the entrepreneur's wealth reaches the outside option. If ever the firm's bank account reaches zero, it must be that the entrepreneur's wealth has reached the outside option as well. By using this assumption the state space is reduced and the contracting problem becomes much more tractable. If we plug both the value of V_2 and the value of W_0 from Equation (26) to the implicit probability of growth from Equation (25) we obtain

$$\tilde{P}_0 \left[\tau_{\bar{k}} < T\ \right] = \frac{K_0 - V_1\left(\frac{q}{1-q}K_0\right)}{(1-q)\,\frac{\mu_{\bar{k}}+\alpha_{\bar{k}}}{r} - V_1\left(\frac{q}{1-q}K_0\right)}. \tag{28}$$

This result takes into account the effect of K_0 on W_0. Based on the technology characteristics, for a given investment amount K_0 we can back out the probability of growth implicit in that investment. Using this result we can compare investments in different technologies and different investment rounds with the same units of account.

Illustrating how this works, suppose for example that the entrepreneur agrees to sell $2/3$ of the firm to investors in $\$13.5M$ valuation: investors commit $K_0 = \$9M$ to the firm and the remaining share belongs to the entrepreneur and equals $\$4.5M$. Figure 5 illustrates that investment of $K_0 = \$9M$ in this specific technology is associated with implicit growth probability of

$$\tilde{P}_0 \left[\tau_{\bar{k}} < T\ \right] \approx \frac{9-1.7}{23\frac{1}{3}-1.7} \approx 33.7\%.$$

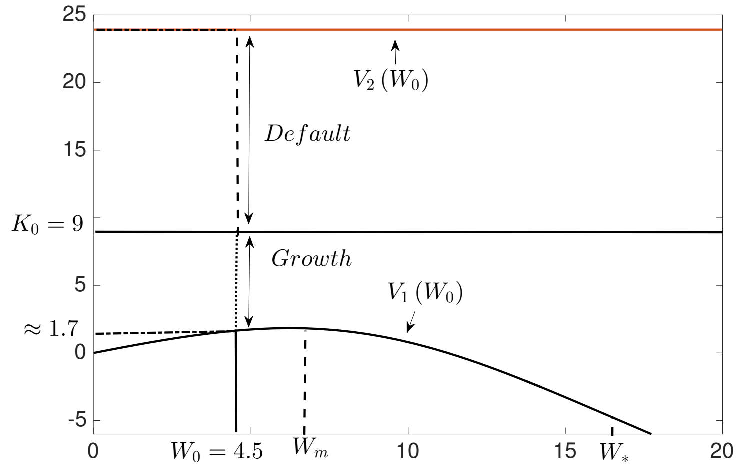


Figure 5. Illustration of the probability of growth associated with investment of $K_0 = 9$. For $L = R = 0$, $\mu_{\bar{k}} = 2$, $\alpha_{\bar{k}} = 1.5$, $q = \frac{1}{3}$, $r = 10\%$, $\gamma = 15\%$, and $\sigma = 4$. W_m is the maximum argument of $V_1(W)$.

5.2 Sensitivity Analysis

In this section we study how the implicit probability of growth changes as we change only one parameter. Our three parameters of interests are: entrepreneur's wealth, initial investment and entrepreneur's equity share, W_0, K_0, q, respectively. As we know from Equation (26), any two parameters determines the third, and given that structure it is not feasible to change only one parameter. Though, it is constructive to break the relationship in Equation (26) and allow only for one parameter to change. To emphasize our exercise we explicitly write the implicit probability of growth as a function of these three parameters, such that $\tilde{P}_0\left(W_0, K_0, q\right)$.

When only entrepreneur's wealth W_0 increases, $\tilde{P}_0\left(W_0 \uparrow, K_0 = b, q = c\right)$, if $W_0 \leq W_m$ ($W_0 > W_m$) the implicit probability of growth decreases (increases). Intuitively, as entrepreneur's wealth approached the outside option the likelihood of early termination increases; however, because the rest of parameters do not change, it corresponds to an increase in the implicit probability of growth. Furthermore, entrepreneur's wealth approached the bonus payment threshold, investors value function decreases; however, because the rest of parameters do not change, it corresponds to an increase in the implicit probability of growth. When only investment K_0 increases, $\tilde{P}_0\left(W_0 = a, K_0 \uparrow, q = c\right)$, the implicit probability of growth increases. Intuitively, investors are willing to invest more capital for the same project. This translates to a higher implicit probability of growth. Lastly, when entrepreneur's equity share increases, $\tilde{P}_0\left(W_0 = a, K_0 = b, q \uparrow\right)$ the implicit probability of may either increase or decrease. An increase in entrepreneur's equity share decrease the value before growth if $W_0 \leq W_m$, but it also decreases the value after growth. This two effects counteract one another. The first decreases and the second increases the implicit probability of growth. The total effect depends on the state. These three effects can be seen geometrically in Figure 6.

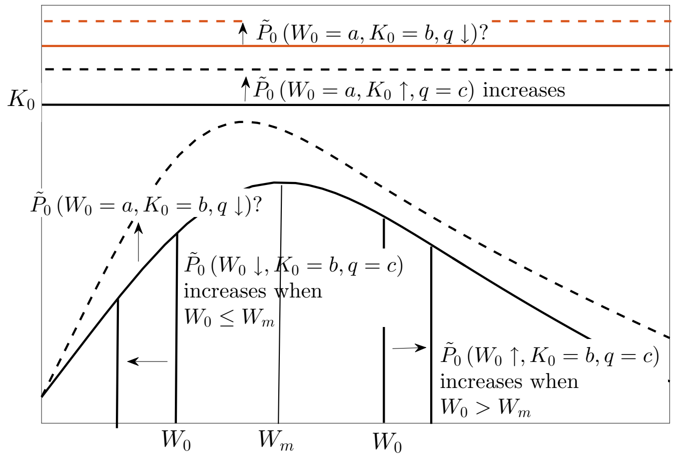


Figure 6. Illustration of the implicit probability of growth sensitivity to changes in three parameters: W_0, K_0, q. W_m is the maximal argument of V_1.

For the following Proposition we introduce the growth likelihood ratio

$$LR(W_0, K_0, q) \equiv \frac{K_0 - V_1\,(W_0)}{V_2\,(W_0) - K_0}. \tag{29}$$

It measures the probability of growth relative to the probability of early termination such that

$$LR(W_0, K_0, q) = \frac{\tilde{P}_0\,(W_0, K_0, q)}{1 - \tilde{P}_0\,(W_0, K_0, q)}.$$

It is well defined for $0 < \tilde{P}_0\,(W_0, K_0, q) < 1$; it goes to $LR(W_0, K_0, q) \uparrow \infty$ as the implicit growth probability approaches 1 and $LR(W_0, K_0, q) \downarrow 0$ as the implicit growth probability approaches 0.

Proposition 5. *Let $\tilde{P}_0\,(W_0, K_0, q)$ represent the implicit probability of growth as function of W_0, K_0, q. If $0 < \tilde{P}_0\,(W_0, K_0, q) < 1$ then*

1. $\frac{\partial \tilde{P}_0(W_0, K_0=b, q=c)}{\partial W_0} \leq 0$ *if* $R \leq W_0 \leq W_m$,
2. $\frac{\partial \tilde{P}_0(W_0, K_0=b, q=c)}{\partial W_0} > 0$ *if* $W_* \geq W_0 > W_m$,
3. $\frac{\partial \tilde{P}_0(W_0=a, K_0, q=c)}{\partial K_0} > 0$,
4. $\frac{\partial \tilde{P}_0(W_0=a, K_0=b, q)}{\partial q} > 0$ *if* $\alpha_{\bar{k}} \geq \mu_{\bar{k}}\left(\frac{1}{LR} - 1\right) + \left(\frac{rL}{1-q}\right)\frac{1}{LR}$,

where W_m is the maximum argument of V_1 and LR is a short for $LR(W_0, K_0, q)$.

The last item in Proposition 5 illustrates that whether the implicit probability increases or decreases depends on the state. In states where the implicit growth probability, $\tilde{P}_0\,(W_0, K_0, q)$, is closer to 1 the likelihood ratio approaches ∞ and the condition in item 4 is easily satisfied; however, in states where the implicit growth probability, $\tilde{P}_0\,(W_0, K_0, q)$, is closer to 0 the likelihood ratio approaches 0 and the condition in item 4 cannot be satisfied.

5.3 Comparative Statics

Our main goal is to compare high-risk and low-risk investments using their implicit probability of growth. Using the analysis in Section 5.2 we now impose the relationship between entrepreneur's wealth, initial investment and entrepreneur's equity share, (W_0, K_0, q), according to Equation (26) and thus properly compare investments.

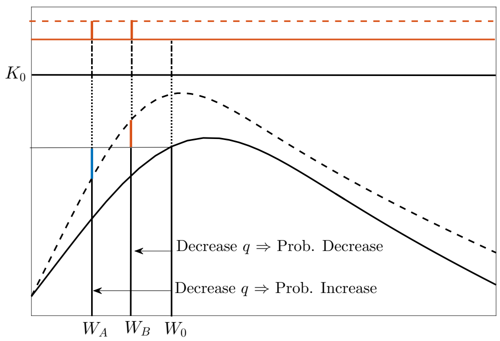
Every investment is determined by two parameters: investors' value and the entrepreneur's value, K_0 and W_0, respectively. In our comparative static analysis we compare investments on two different dimensions. In the first dimension we keep investors' valuation fixed and change the entrepreneur's valuation; in the second dimension we keep the entrepreneur's valuation fixed and change investors' valuation. Our analysis shows that contrary to common perception that higher valuation and higher investments translate to lower risk this is not always the case.

In the first analysis, when W_0 decreases and K_0 remains fixed, which implies that investors equity share, $1-q$, increases. In this case, there are two forces at play: First, both the value functions V_1 and V_2 increase [6]. The increase in the value function before growth implies that the implicit probability of growth decreases while the increase in the value function after growth implies that the implicit probability of growth increases. The total effect depends on the state; for $W_0 \leq W_m$, the closer W_0 is to R the positive effect is stronger and the implicit probability of growth eventually increases. Second, the entrepreneur's wealth decreases. This implies that the probability of early termination is higher. But because K_0 is fixed this translates to higher implicit growth probability. Figure 7a shows the two forces. When W_0 decreases all the way to W_A, it is the low entrepreneur's wealth effect that dominates. In this case, the overall effect translates to higher implicit growth probability. This is contrary to common perception that lower valuation implies lower growth probability. However, when W_0 decreases to W_B, it is the higher value function effect that dominates. In this case, the overall effect translates to lower implicit growth probability.

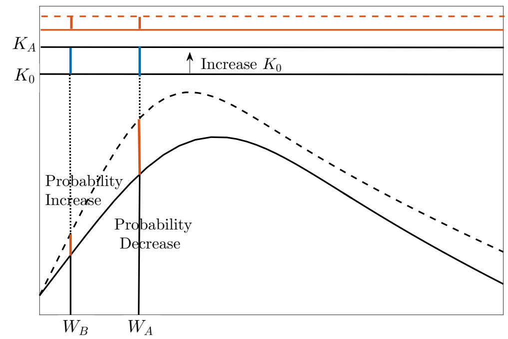
In the second analysis, when K_0 increases and W_0 remains fixed, investors equity share, $1-q$, increases as well, Equation (26). In this case, there are two forces at play: First, similar to the previous analysis, the value functions V_1 and V_2 increase. The increase in the value function before growth implies that the implicit probability of growth decreases while the increase in the value function after growth implies that the implicit probability of growth increases, and similar to the previous analysis the total effect is positive if W_0 is sufficiently close to R. Second, initial investment increases, which translates to higher implicit growth probability. Figure 7b shows the two opposing forces. When the entrepreneur's wealth is fixed as high as W_A, it is the higher value function effect that dominates. This is also contrary to common perception that higher investment implies higher implicit growth probability. However, when the entrepreneur's wealth is fixed as low as W_B, it is the higher initial investment effect that dominates. In this case, the overall effect is higher implicit

[6] According to Proposition 5 V_1 increases when $W_0 \leq W_m$.

growth probability.



(a) A static comparison of implicit probability of growth when K_0 is fixed. W_0 falls to either W_A or W_B. Due to changes in the equity share q, when W_0 falls to W_A, the implicit growth probability increases, and when W_0 falls to W_B, the implicit growth probability decreases.



(b) A static comparison of implicit probability of growth when W_0 is fixed either to W_A or W_B. Due to changes in equity share q, when entrepreneur's wealth is W_A, an increase in the initial investment, K_0 translates to a decrease in the implicit growth probability, and when the entrepreneur's wealth is W_B, an increase in the initial investment, K_0, translates to an increase in the implicit growth probability.

Figure 7. Comparative Statics. In both Figures the probability of growth may increase or decrease depending on parametrization. The red and blue sticks in the Figures signifies a decrease and an increase in the implicit probability of growth, respectively.

The Figures 7a and 7b shows that, in any case, the total effect of changes to investment on the implicit growth probability is highly susceptible to the parameter setting. Changes to one parameter changes the others as well.

5.4 Investment Rounds

When the entrepreneur's wealth reaches the outside option, as long as investors' assessment of the probability of growth is strictly positive, both the entrepreneur and investors would find it optimal to jump-start the firm with an additional capital injection and with entrepreneur's value $W > R$. We can interpret these capital injections as new investment rounds. Each new investment adds a lump sum capital to the bank balance ΔK_t, in exchange for an additional share Δq. The level of capital $K_t + \Delta K_t$ would implicitly determine investors implicit probability of growth and the firms would start afresh at the new point $(W_t, K_t + \Delta K_t)$ with the new equity share $q - \Delta q$.

Intuitively, firm's value is a weighted average of before and after growth materializes value functions. As long as investors assessment of the probability of growth is strictly positive it increases firm's value to inject additional capital in exchange for an additional equity piece in the firm. This is in contrast to DeMarzo and Sannikov (2006) showing that when $V_1'(R) > 0$, starting afresh with new initial wealth W is beneficial as long as $V_1(W) > V_1(R)$. Although, in a broader perspective. From an ex-post perspective every additional investment round decreases the return on the total investment. This would reduce the attractiveness of an additional investment round compared to other potential investments.

5.5 Further Discussion

How to motivate entrepreneurs to pursue high-growth? In practice, once a suitable project is found, a short document with all the pertinent information — a term-sheet — is signed. Term sheets are usually not public, but they are comprised of standard mechanisms to achieve investors' goals. In particular aligning incentives to pursue high-growth. One standard mechanism that motivates the entrepreneur to pursue high growth is by issuing preferred stocks to investors as opposed to common stocks to the entrepreneur. Preferred stocks have a higher priority, which means investors receive their initial investment plus an additional predetermined return before any allocation is made to common stock holders — the entrepreneur. An even more powerful mechanism being employed regularly is commonly referred as 'double deep'. Intuitively, double deep means that investors' preferred stocks participate also as common stocks: after getting paid as preferred stock holders, investors participate again as common stock holders.

Though, intended as risk reduction techniques for investors, in essence, these mechanisms and others like them provide a lower bound on potential returns: any project that does not have the potential to deliver returns higher than the lower bound is not worth pursing. In turn, it forces entrepreneurs to pursue projects with high growth. However, setting the appropriate boundary is not straight forward. A proper lower-bound should separate low- from high- growth technologies. But in order to do it investors have to know about all the competing technologies and their potential growth, which is practically impossible in many cases. In this paper we show how to motivate the entrepreneur to pursue high-growth with minimal, or even no, knowledge about competing technologies.

6 Extensions

We introduce two extensions to the main economic setup. In the first extension, instead of having two technologies, we increase the set of possible technologies. In the second extension, instead of arriving at once, we allow growth to arrive incrementally over time.

6.1 The General Economic Setup

In this section, we accommodate a market with multiple technologies each with a different growth opportunity. Denote by $\mathcal{K}$ the set of possible technologies; each technology produces cash-flow characterized by

$$dX_{kt} = \Big(\mu_k + \alpha_k \mathbf{1}_{\tau_k}\Big) dt + \sigma dZ_t, \quad \forall k \in \mathcal{K}. \tag{30}$$

Let $\bar{k} \in \mathcal{K}$ be the technology with the highest growth, such that,

$$\alpha_{\bar{k}} > \alpha_k, \ \forall k \in \mathcal{K}, \neq \bar{k}. \tag{31}$$

Agency Friction:
By switching from lower-growth technology to the highest-growth technology the firm forgoes higher immediate expected cash-flow,

$$\mu_{\bar{k}} - \mu_k < 0, \ \forall k \in \mathcal{K}, \neq \bar{k}, \tag{32}$$

in exchange for a risky gamble on the highest-growth in the future. If the highest growth materializes, the firm's expected cash-flow increases substantially,

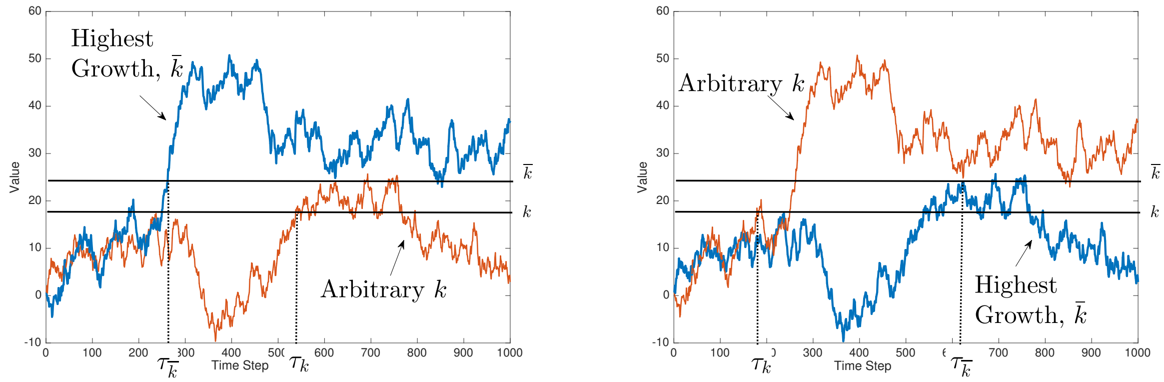
$$\mu_{\bar{k}} + \alpha_{\bar{k}} > \mu_k + \alpha_k, \ \forall k \in \mathcal{K}, \neq \bar{k}. \tag{33}$$

Investors prefer a risky investment profile but do not have the skill set to implement the new high-growth technology — alternatively, running the firm requires too much time and effort that is efficiently allocated to different tasks. For that they need the entrepreneur; however, the entrepreneur prefers early payments and motivating her to postpone payments and pursue the highest growth requires sufficient incentives.

Information:
Investors observe two pieces of information: the cash-flow reported by the entrepreneur

$$dY_t = dX_{kt} \quad \exists k \in \mathcal{K}, \quad \forall t \in [0, T], \tag{34}$$

and the growth arrival time $1_{\tau_{\bar{k}}}$. The entrepreneur decides on which technology to implement; this decision is private and is not observed by investors. Furthermore, similar to investors, the entrepreneur observes $1_{\tau_{\bar{k}}}$.



(a) An example of two realizations where the highest growth technology materializes *before* a lower growth technology.

(b) An example of two realizations where the highest growth technology materializes *after* a lower growth technology.

Figure 8. Modeling growth arrival times using barriers with independent $\tilde{Z}_{kt}, \forall k \in \mathcal{K}$. The blue wider line is a realization of the highest-growth technology, $\bar{k}$, and the red thiner line is a realization of an arbitrary technology, $k \in \mathcal{K}, \neq \bar{k}$.

Under the above assumptions, the results from the main section carry over. The following Proposition summarizes this section result.

Proposition 6. *The highest growth strategy $\bar{k}$ is incentive compatible if and only if both conditions*

1. $\beta_t \leq 0$, *when* $\mathbf{1}_{\tau_{\bar{k}}} = 0$
2. $\beta_t \geq 0$, *when* $\mathbf{1}_{\tau_{\bar{k}}} = 1$

are satisfied for all $t \leq T$. And the optimal contract from Proposition 1 carries over.

6.2 Incremental Growth

In previous sections, growth arrives in a single time. In this section, we mitigate that assumption by allowing growth to arrive incrementally over time. For simplicity, we assume that growth arrives in two increments. Accordinlgy, each technology produces cash-flow characterized by

$$dX_{kt} = (\mu_k + \alpha_{1k}\mathbf{1}_{\tau_{1k}} + \alpha_{2k}\mathbf{1}_{\tau_{2k}})\, dt + \sigma dZ_t, \quad \forall k \in \mathcal{K}. \tag{35}$$

Let $\bar{k} \in \mathcal{K}$ be the technology with the highest growth. Such that,

$$\alpha_{1\bar{k}} + \alpha_{2\bar{k}} > \alpha_{1k} + \alpha_{2k}, \ \forall k \in \mathcal{K}, \neq \bar{k}. \tag{36}$$

Agency Friction:

By switching from lower-growth technology to the highest-growth technology the firm forgoes higher immediate expected cash-flow,

$$\mu_{\bar{k}} + \alpha_{i\bar{k}} < \mu_k, \ \forall k \in \mathcal{K}, \neq \bar{k}, \ i = 1, 2, \tag{37}$$

in exchange for a risky gamble on the highest-growth. If the highest growth materializes, the firm's expected cash-flow increases substantially,

$$\mu_{\bar{k}} + \alpha_{1\bar{k}} + \alpha_{2\bar{k}} > \mu_k + \alpha_{1k} + \alpha_{2k}, \; \forall k \in \mathcal{K}, \neq \bar{k}. \tag{38}$$

As in all previous sections, investors prefer a risky investment profile but do not have the skill set to implement the new high-growth technology. For that they need the entrepreneur; however, the entrepreneur prefers early payments and motivating her to postpone payments and pursue the highest growth requires sufficient incentives.

Information: Investors observe three pieces of information: the cash-flow reported by the entrepreneur

$$dY_t = dX_{kt} \quad \exists k \in \mathcal{K}, \quad \forall t \in [0, T], \tag{39}$$

and the growth arrival times $1_{\tau_{1\bar{k}}}$ and $1_{\tau_{2\bar{k}}}$. The entrepreneur decides on which technology to implement; this decision is private and is not observed by investors. Furthermore, similar to investors, the entrepreneur observes the growth arrival times.

With incremental growth there are three value functions. Before growth, after the first growth, and after the second growth. Due to agency friction, as long as the second growth has not arrived, the condition in Equation (37) holds. This implies that the first two value functions have a similar form. Figure 9 illustrates these two value functions.

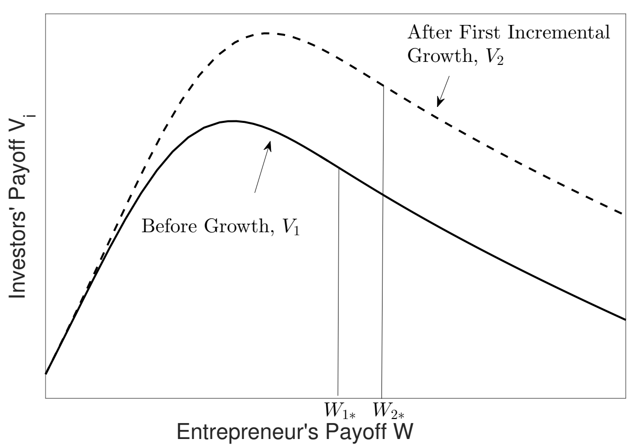


Figure 9. Optimal contract with incremental growth.

Under the above assumptions, the results from the main section carry over with minor modifications to accommodate two stages of growth. The following Proposition summarizes this section result.

Proposition 7. *The highest growth strategy $\bar{k}$ is incentive compatible if and only if both conditions*

1. $\beta_t \geq 0,$ *when* $\mathbf{1}_{\tau_{1\bar{k}}} = \mathbf{1}_{\tau_{2\bar{k}}} = 1$
2. $\beta_t \leq 0,$ *otherwise*

are satisfied for all $t \leq T$. During the implementation phase ($t < \tau_{1\bar{k}}$) W_t evolves according to

$$dW_t = \gamma W_t dt - q\mu_{\bar{k}} dt - q\left(dY_t - \mu_{\bar{k}} dt\right) \tag{40}$$

when $W_t \in [R, W_{1*}]$ and $dI_t = 0$. When $W_t > W_{1*}$ a cash payments dI_t is made so that $W_t = W_{1*}$. Investors expected payoff is concave and follows the ordinary differential equation given in Equation (9) subject to the smooth pasting, super-contract and initial conditions given in Equations (10), (11) and (12) at W_{1*}; the contract is terminated when W_t reaches R denoted by time T.
After the first incremental growth ($\tau_{1\bar{k}} \leq t < \tau_{2\bar{k}}$) W_t evolves according to

$$dW_t = \gamma W_t dt - (1-q)\, dI_t - q\left(\mu_{\bar{k}} + \alpha_{1\bar{k}}\right) dt + (\beta_t - q)\Big(dY_t - \left(\mu_{\bar{k}} + \alpha_{1\bar{k}}\right) dt\Big) \tag{41}$$

when $W_t \in [R, W_{2*}]$ and $dI_t = 0$. When $W_t > W_{2*}$ a cash payments dI_t is made so that $W_t = W_{2*}$. Investors expected payoff is concave and follows the ordinary differential equation given by

$$rV_2 = (1-q)\left(\mu_{\bar{k}} + \alpha_{1\bar{k}}\right) + \Big(\gamma W - q\left(\mu_{\bar{k}} + \alpha_{1\bar{k}}\right)\Big) V_2' + \frac{1}{2}\sigma^2 q^2 V_2'', \tag{42}$$

in Equation (9) subject to the smooth pasting, super-contract and initial conditions given in Equations (10), (11) and (12) at W_{2*} ; the contract is terminated when W_t reaches R denoted by time T. After the second incremental growth ($t \geq \tau_{2\bar{k}}$) investors expected payoff reaches the first-best absorbing state

$$V_3(W) = v_3 \equiv (1-q)\frac{\mu_{\bar{k}} + \alpha_{1\bar{k}} + \alpha_{2\bar{k}}}{r}, \qquad W_t = w_2 \equiv q\frac{\mu_{\bar{k}} + \alpha_{1\bar{k}} + \alpha_{2\bar{k}}}{r}. \tag{43}$$

7 Conclusion

The risk associated with start-up investments is difficult to measure because the usual risk measures do not exist. However, by investing in these firms investors de-facto assess the risk of an early-stage firm relative to other investments. Our main goal is to measure that price-in risk; we do so by introducing the implicit probability of growth. By measuring the relative risk bore by investors in general and by VC's (venture capitalists) in particular we can study the risk-profile both across and within VC's.

In order to derive the implicit probability of growth, we study a principal-agent problem in a continuous time, dynamic setting with modifications to accommodate the tradeoff between future high-growth and the short-term lack of performance. The agency friction arises because the entrepreneur has a shorter time-horizon; thus, motivation to pursue high-growth requires incentives. In the optimal contract, the entrepreneur's desired actions have an adverse effect on the near-term performance of the firm in exchange for high-growth potential in the future. This is achieved by transferring bonus payments after histories of poor performance. The contract is implemented by introducing a limited liability firm that is jointly owned by investors and the entrepreneur.

Previous literature on early-stage firms and how to motivate innovation in the principal-agent context was focused on a static, two-period framework. In spite the fact that the empirical literature emphasizes the dynamic nature of this problem. We further contribute to the literature by introducing a persistent friction between the parties that lasts over an extended and random amount of time. Our result is inline with previous literature and more specifically with Manso (2011).

Our model can be naturally extended to accommodate long-term incentives in different applications. For example, a manager that faces a tradeoff between fast implementation of some task with an old employee or slow implementation with a new employee. The tradeoff is between implementation time and future capacity. Adding an additional employee increases the future capacity but also undesirably increases the implementation time. A manager with a short-horizon may prefer a short implementation time on high future capacity that may be more important to a principal with long-term objectives.

Appendix A: Proofs

Proof of Lemma 1 . Let us conjecture that $\int_0^T e^{-\gamma s}\Big\{qdY_s + (1-q)\,dI_s\Big\}$ is uniformly bounded, which can be verified once the optimal contract is obtained. The process

$$W_t\left(Y, \tau_{\bar{k}}\right) e^{-\gamma t} + \int_0^t e^{-\gamma s}\Big\{qdY_s + (1-q)\,dI_s\Big\}$$

is a martingale for every $t \leq T$, with respect to the filtration generated by Z_t. By the Martingale Representation Theorem it can be written as a stochastic integral,

$$\int_0^t \bar{\beta}_s dZ_s.$$

Define $\beta_t \equiv \frac{\bar{\beta}_t}{\sigma e^{-\gamma t}}$; after using Itô's Lemma we obtain

$$\begin{aligned} dW_t &= \gamma W_t dt - (1-q)\,dI_t - qdY_t + \beta_t \sigma dZ_t \\ &= \gamma W_t dt - (1-q)\,dI_t - q\left(\mu_{\bar{k}} + \alpha_{\bar{k}}\mathbf{1}_{\tau_{\bar{k}}}\right)dt + (\beta_t - q)\,\sigma\Big(dX_{\bar{k}} - \left(\mu_{\bar{k}} + \alpha_{\bar{k}}\mathbf{1}_{\tau_{\bar{k}}}\right)dt\Big), \end{aligned}$$

which leads to the desired result. □

Proof of Lemma 2 . Suppose the entrepreneur follows the zero growth strategy $\underline{k} \neq \overline{k}$ up to time t and after time t moves to $\overline{k}$. The total expected payoff from following this strategy is

$$X_t = \int_0^t e^{-\gamma s}\Big(\,(1-q)\,dI_s + qdY_s\Big) + e^{-\gamma t}W_t.$$

By Itô's Lemma

$$\begin{aligned} e^{\gamma t}dX_t &= (1-q)\,dI_t + q\mu_{\underline{k}}dt + q\left(dY_t - \mu_{\underline{k}}dt\right) \\ &\quad - \gamma W_t dt + \gamma W_t dt - (1-q)\,dI_t - q\left(\mu_{\bar{k}} + \alpha_{\bar{k}}\mathbf{1}_{\tau_{\bar{k}}}\right)dt + (\beta_t - q)\left(dY_t - \mu_{\bar{k}}dt - \alpha_{\bar{k}}\mathbf{1}_{\tau_{\bar{k}}}dt\right) \\ &\quad = q\mu_{\underline{k}}dt + q\left(dY_t - \mu_{\underline{k}}dt\right) - q\left(\mu_{\bar{k}} + \alpha_{\bar{k}}\mathbf{1}_{\tau_{\bar{k}}}\right)dt \\ &\quad + (\beta_t - q)\Big[\mu_{\underline{k}} - \left(\mu_{\bar{k}} + \alpha_{\bar{k}}\mathbf{1}_{\tau_{\bar{k}}}\right)\Big]dt + (\beta_t - q)\left(dY_t - \mu_{\underline{k}}dt\right) \\ &= \beta_t\Big[\mu_{\underline{k}} - \left(\mu_{\bar{k}} + \alpha_{\bar{k}}\mathbf{1}_{\tau_{\bar{k}}}\right)\Big]dt + \beta_t\left(dY_t - \mu_{\underline{k}}dt\right) \end{aligned}$$

Suppose $\mathbf{1}_{\tau_{\bar{k}}} = 0$: If $\beta_t \leq 0$ the value process X_t has a non-positive drift,

$$\beta_t\left[\mu_{\underline{k}} - \mu_{\bar{k}}\right] \leq 0. \tag{A.1}$$

If there is a positive probability event that $\beta_t > 0$ for $t < T$, by implementing $\underline{k}$ the entrepreneur is better off.
Suppose $\mathbf{1}_{\tau_{\bar{k}}} = 1$: If $\beta_t \geq 0$ the value process X_t has a non-positive drift,

$$\beta_t\Big[\mu_{\underline{k}} - \left(\mu_{\bar{k}} + \alpha_{\bar{k}}\right)\Big] \leq 0 \tag{A.2}$$

If there is a positive probability event that $\beta_t < 0$ for $t < T$, by implementing $\underline{k}$ the entrepreneur is better off. Thus, implementing $\bar{k}$ is incentive compatible if and only if both conditions in items 1 and 2 hold.

$\square$

Proof of Proposition 1 . Let us first discuss the value function before growth matures. Similar to DeMarzo and Sannikov (2006), the value function is concave. This can be seen by plugging the conditions $V_1'(W) \geq -1$ and $rV_1(W) < \mu_{\bar{k}} - \gamma W$ into the HJB Equation (9). This leads to $V_1'' < 0$ on $[0, W_*)$, and by construction, at the boundary, $V_1''(W_*) = 0$. Showing that it is optimal, let us define the gain process

$$G_t \equiv \int_0^t e^{-rs}\Big\{ (1-q)\,(dY_s - dI_s) \Big\} + e^{-rt}\Big[V_1\,(W)\,\big(1 - \mathbf{1}_{\tau_{\bar{k}}}\big) + V_2\,(W)\,\big(\mathbf{1}_{\tau_{\bar{k}}}\big)\Big].$$

To analyze the case where growth did not materialize, we impose that $\mathbf{1}_{\tau_{\bar{k}}} = 0$ and by Itô's Lemma, under any incentive compatible contract and for given W_t, which evolves according to Equation (8), we have

$$e^{rt} dG_t = \underbrace{\Big((1-q)\,\mu_{\bar{k}} + (\gamma W_t - q\mu_{\bar{k}})\,V_1' + \frac{1}{2} V_1''\,(\beta_t - q)^2\,\sigma^2 - rV_1 \Big)}_{\leq 0} dt + (1-q)\,\underbrace{\big(-1 - V_1'\big)}_{\leq 0} dI_t$$
$$+ \underbrace{\Big((1-q) + V_1'\,(\beta_t - q) \Big)\sigma dZ_t}_{\text{martingale}}.$$

The dt term is non-positive because $V_1'' < 0$ and $V_1' \geq -1$. The dI_t term is non-positive because $V_1' \geq -1$. Under the optimal policy, due to concavity $\beta_t = 0$, the dt term becomes 0 and the payout policy is such that $dI_t = 0$ whenever $V_1' \geq -1$ and $dI_t > 0$ whenever $V_1' = -1$. Under the optimal policy, the last element is a martingale because V_1' is uniformly bounded and $\beta_t = 0$. Investors payoff under any incentive compatible contract is

$$\begin{aligned}
&E\left[\int_0^T e^{-rs}\Big\{ (1-q)\,\Big(dY_s - dI_s\Big)\Big\} + e^{-rT}L\right]\\
=&E\left[G_{t\wedge T} + \mathbf{1}_{t\leq T}\left(\int_t^T e^{-rs}\Big\{ (1-q)\,\Big(dY_s - dI_s\Big)\Big\} - e^{-rt}V_1(W_t) + e^{-rT}L\right)\right]\\
=&E\left[G_{t\wedge T}\right] + e^{-rt}E\left[\mathbf{1}_{t\leq T}\left(\int_t^T e^{-r(s-t)}\Big\{ (1-q)\,\Big(dY_s - dI_s\Big)\Big\} + e^{-r(T-t)}L - V_1(W_t)\right)\right]\\
\leq&V_0(W_0) + e^{-rt}\,(1-q)\,\frac{\mu_{\bar{k}}}{r}.
\end{aligned}$$

In the last inequality, the first term is due to the super-martingale property of G_t under any incentive compatible contract; the second term is the first best solution. It is investors' share of all the future income when the contract is never terminated and no payments are made to the entrepreneur. Letting $t \to \infty$ we obtain

$$E\left[\int_0^T e^{-rs}\Big\{ (1-q)\,\Big(dY_s - dI_s\Big)\Big\} + e^{-rT}L\right] \leq V_0(W_0).$$

The super-martingale becomes a martingale under the optimal contract and the weak inequality becomes an equality.
Let us now discuss the case where growth materialized. For that we impose that $\mathbf{1}_{\tau_{\bar{k}}} = 1$ and by

Itô's Lemma, under any incentive compatible contract and for a given W_t, which evolves according to Equation (8), we have

$$e^{rt}dG_t = \Big((1-q)\left(\mu_{\bar{k}} + \alpha_{\bar{k}}\right) + \left(\gamma W_t - q\left(\mu_{\bar{k}} + \alpha_{\bar{k}}\right)\right) V_2' + \frac{1}{2} V_2'' \left(\beta_t - q\right)^2 \sigma^2 - rV_2 \Big) dt + (1-q)\left(-1 - V_2'\right) dI_t$$
$$+ \Big((1-q) + V_2' \left(\beta_t - q\right) \Big) \sigma dZ_t.$$

Under any incentive compatible contract, due to concavity $\beta_t = q$. This implies that the first-best solution can be obtained. Under this policy, there are no transfers from investors to the entrepreneur, $dI_t = 0$, and they both consume their own share of the pie indefinitely,

$$V_2 = (1-q)\frac{\mu_{\bar{k}} + \alpha_{\bar{k}}}{r}, \qquad W_t = q\frac{\mu_{\bar{k}} + \alpha_{\bar{k}}}{\gamma}.$$

□

Proof of Proposition 2 . Before growth materializes and when $q = 0$ the optimal contract solves

$$rV_1 = \mu_{\bar{k}} + \gamma W V_1'. \tag{A.3}$$

The solution to this equation is

$$V_1 = \frac{\mu_{\bar{k}}}{r} + c\left(\gamma W\right)^{\frac{r}{\gamma}},$$

where c is a Real constant. The validity of the solution is easily verified by plugging it back to Equation (A.3). If we derivate this equation twice we get that

$$V_1'' = cr\left(r - \gamma\right)\left(\gamma W\right)^{\frac{r}{\gamma} - 2}. \tag{A.4}$$

Because $r \le \gamma$, concavity of the solution implies that $c \ge 0$, and existence implies that $W > 0$. Furthermore, the optimal contract cannot achieve higher expected return than total expected payoff of running the project indefinitely, $\frac{\mu_{\bar{k}}}{r}$, which implies that $c = 0$. However, if $c = 0$ investors rake all the future value from the project which implies that $W = 0$, which implies an immediate termination, because $R \ge 0$. □

Proof of Proposition 3 . First, to show that on-time disclosure is better than overdue disclosure we claim that

$$W_*(q) \ge q\frac{\mu_{\bar{k}}}{\gamma}$$

for any $1 > q > 0$. When there is no agency friction, the parties split the total surplus between them relative to their equity share, with the justification similar to the proof of Proposition 1 in the after-growth case. At W_* we are on the line that defines the total surplus; however, due to the agency friction investors have to transfer funds to align incentives. This implies that at that point, the entrepreneur's total wealth is larger than her equity share and investors total wealth is lower than their equity share. Using this result, for a given set of parameters the growth potential parameter $\alpha_{\bar{k}}$ determines whether the Inequality (15) holds, and for sufficiently large growth potential this inequality always holds.
Second, we show that on-time disclosure is better than premature disclosure. The key argument for this part is that as $W_t \downarrow R$ the probability of termination before growth goes up as well

$P_t\left[\tau_{\bar{k}} < T\right] \downarrow 0$. This is because we assume independence between growth arrival $\tau_{\bar{k}}$ and the cash-flow shocks, dZ_t that determines the early termination time T. Thus,

$$W_t \downarrow R \Longrightarrow P_t\left[\tau_{\bar{k}} < T\right] \downarrow 0 \tag{A.5}$$

Let $R > 0$. ($\Rightarrow$) Suppose that by contradiction Inequality (17) does not hold . Let us take a history that sends W_t sufficiently close to R, $|W_t - R| < \epsilon$ for an arbitrarily small ϵ. The probability of termination is sufficiently high so that

$$q\frac{\mu_{\bar{k}}}{\gamma} > P_t\left[\tau_{\bar{k}} < T\right]\left\{q\frac{\mu_{\bar{k}} + \alpha_{\bar{k}}}{\gamma}\right\} + \Big(1 - P_t\left[\tau_{\bar{k}} < T\right]\Big)W_t \tag{A.6}$$

In this case, by falsely reporting growth, the entrepreneur obtains the left hand side of Inequality (A.6), which is clearly better a strategy. A contradiction to truthful reporting.
($\Leftarrow$) Let us assume by contradiction that falsely reporting growth is an optimal strategy. In that case her wealth is set at $q\frac{\mu_{\bar{k}}}{\gamma}$; however, $R > q\frac{\mu_{\bar{k}}}{\gamma}$, which determines this strategy sub-optimal. A contradiction. $\square$

Proof of Proposition 4 . From Equations (8) and (19) we observe that when there is a payment dI_t, a fraction $1-q$ is attributed to W_t and a fraction q is attributed to qK_t. Furthermore, the interest rate accumulates both on qK_t and W_t with the same rate. This implies that in-between payments both the wealth and q shares of the bank account increases proportionally; thus, during these times, c_t moves parallel to the line between the origin and (W_0, qK_0) on the state space plane. For a given initial point (W_0, qK_0) the relative equity share size determines whether dI_t payments induce c_t to be above or below the line connecting the origin and the initial point (W_0, qK_0). This is determined by comparing the angles of $\tan^{-1}\left(\frac{W_0}{qK_0}\right)$ and $\tan^{-1}\left(\frac{1-q}{q}\right)$, as can be seen geometrically in Figure 4. Requiring to be below the dashed line implies that

$$W_0 \leq (1-q)\,K_0.$$

$\square$

Proof of Proposition 5 . The implicit probability of default is given in Equation (25). It is immediate to see that $\frac{\partial \tilde{P}_0(W_0=a,K_0,q=c)}{\partial K_0} > 0$. Furthermore, according to Proposition 1, from the concavity of V_1, we know that $\frac{\partial V_1(W_0)}{\partial W_0} \geq 0$ for $R \leq W_0 \leq W_m$ and $\frac{\partial V_1(W_0)}{\partial W_0} < 0$ for $W_* > W_0 > W_m$, and also that V_2 is independent of W_0. Therefore, when $V_2 > K_0 > V_1$, which is guaranteed when under our assumption that $0 < \tilde{P}_0(W_0, K_0, q) < 1$, we obtain $\frac{\partial \tilde{P}_0(W_0,K_0=b,q=c)}{\partial W_0} \leq 0$ for $R \leq W_0 \leq W_m$ and $\frac{\partial \tilde{P}_0(W_0,K_0=b,q=c)}{\partial W_0} > 0$ for $W_* > W_0 > W_m$.
To find $\frac{\partial \tilde{P}_0(W_0=a,K_0=b,q)}{\partial q}$ we have to investigate both $\frac{\partial V_1(W_0)}{\partial q}$ and $\frac{\partial V_2(W_0)}{\partial q}$. It is immediate to see that $\frac{\partial V_2(W_0)}{\partial q} = -\frac{\mu_{\bar{k}}+\alpha_{\bar{k}}}{r}$. Lastly, to investigate $\frac{\partial V_1(W_0)}{\partial q}$ we follow the comparative statics results in DeMarzo and Sannikov (2006). Applying their result to our formulation of V_1 in Equation (9) we find that

$$\frac{\partial V_1(W)}{\partial q} = E\left[\int_0^T e^{-rt}\Big[\underbrace{\Big(-\mu_{\bar{k}}\left(1+V_1'(W)\right) + \sigma^2 q V_1''(W)\Big)}_{\leq 0}dt + \underbrace{dI_t}_{\geq 0}\Big] \mid W_0 = W\right]. \tag{A.7}$$

From Equation (6) we know that

$$E\left[\int_0^T e^{-rt} dI_t\right] = -\frac{V_1(W_0)}{1-q} + \frac{\mu_{\bar{k}}}{r} E\left[\left(1-e^{-rT}\right)\right] + E\left[e^{-rT}\frac{L}{1-q}\right]. \tag{A.8}$$

Combining these two equations together we obtain

$$\frac{\partial V_1(W)}{\partial q} \le \frac{\mu_{\bar{k}}}{r} E\left[\left(1-e^{-rT}\right)\right] + E\left[e^{-rT}\frac{L}{1-q}\right] \le \frac{\mu_{\bar{k}}}{r} + \frac{L}{1-q}. \tag{A.9}$$

It can be easily verified geometrically (Figure 5) that

$$\frac{\partial LR\left(W_0 = a, K_0 = b, q\right)}{\partial q} \ge 0 \iff \frac{\partial \tilde{P}_0\left(W_0 = a, K_0 = b, q\right)}{\partial q} \ge 0$$

and that

$$\frac{\partial LR\left(W_0 = a, K_0 = b, q\right)}{\partial q} \ge 0 \iff -\frac{\partial V_1(W)}{\partial q} + LR\left(W_0 = a, K_0 = b, q\right)\frac{\mu_{\bar{k}} + \alpha_{\bar{k}}}{r} \ge 0$$

when $1 > \tilde{P}_0\left(W_0, K_0, q\right) > 0$. This inequality holds if

$$LR\left(W_0 = a, K_0 = b, q\right)\frac{\mu_{\bar{k}} + \alpha_{\bar{k}}}{r} \ge \frac{\mu_{\bar{k}}}{r} + \frac{L}{1-q} \ge \frac{\partial V_1(W)}{\partial q}.$$

Impose $\alpha_{\bar{k}}$ such that

$$\alpha_{\bar{k}} \ge \mu_{\bar{k}}\left(\frac{1}{LR\left(W_0 = a, K_0 = b, q\right)} - 1\right) + \left(\frac{rL}{1-q}\right)\frac{1}{LR\left(W_0 = a, K_0 = b, q\right)}$$

and obtain our desired result.

$\square$

Proof of Proposition 6 . The incentive compatibility proof is slightly adjusted to accommodate a bigger set of technologies. Suppose the entrepreneur follows a strategy $k \neq \overline{k}$ up to time t and after time t moves to $\overline{k}$. The total expected payoff from following this strategy is

$$X_t = \int_0^t e^{-\gamma s}\Big(\left(1-q\right)dI_s + q dY_s\Big) + e^{-\gamma t} W_t.$$

By Itô's Lemma

$$\begin{aligned}
e^{\gamma t} dX_t = {} & (1-q)\, dI_t + q\left(\mu_k + \alpha_k \mathbf{1}_{\tau_k}\right) dt + q\left(dY_t - \left(\mu_k + \alpha_k \mathbf{1}_{\tau_k}\right) dt\right) \\
& - \gamma W_t dt + \gamma W_t dt - (1-q)\, dI_t - q\left(\mu_{\bar{k}} + \alpha_{\bar{k}} \mathbf{1}_{\tau_{\bar{k}}}\right) dt + (\beta_t - q)\left(dY_t - \mu_{\bar{k}} dt - \alpha_{\bar{k}} \mathbf{1}_{\tau_{\bar{k}}} dt\right) \\
& = q\left(\mu_k + \alpha_k \mathbf{1}_{\tau_k}\right) dt + q\left(dY_t - \left(\mu_k + \alpha_k \mathbf{1}_{\tau_k}\right) dt\right) - q\left(\mu_{\bar{k}} + \alpha_{\bar{k}} \mathbf{1}_{\tau_{\bar{k}}}\right) dt \\
& + (\beta_t - q)\Big[\left(\mu_k + \alpha_k \mathbf{1}_{\tau_k}\right) - \left(\mu_{\bar{k}} + \alpha_{\bar{k}} \mathbf{1}_{\tau_{\bar{k}}}\right)\Big] dt + (\beta_t - q)\left(dY_t - \left(\mu_k + \alpha_k \mathbf{1}_{\tau_k}\right) dt\right) \\
= {} & \beta_t\Big[\left(\mu_k + \alpha_k \mathbf{1}_{\tau_k}\right) - \left(\mu_{\bar{k}} + \alpha_{\bar{k}} \mathbf{1}_{\tau_{\bar{k}}}\right)\Big] dt + \beta_t\left(dY_t - \left(\mu_k + \alpha_k \mathbf{1}_{\tau_k}\right) dt\right)
\end{aligned}$$

Suppose $\mathbf{1}_{\tau_{\bar{k}}} = 0$: If $\beta_t \leq 0$ the value process X_t has a non-positive drift,

$$\beta_t \Big[(\mu_k + \alpha_k \mathbf{1}_{\tau_k}) - \mu_{\bar{k}} \Big] \leq 0. \tag{A.10}$$

If there is a positive probability event that $\beta_t > 0$ for $t < T$, by implementing k the entrepreneur is better off.
Suppose $\mathbf{1}_{\tau_{\bar{k}}} = 1$: If $\beta_t \geq 0$ the value process X_t has a non-positive drift,

$$\beta_t \Big[(\mu_k + \alpha_k \mathbf{1}_{\tau_k}) - (\mu_{\bar{k}} + \alpha_{\bar{k}}) \Big] \leq 0. \tag{A.11}$$

If there is a positive probability event that $\beta_t < 0$ for $t < T$, by implementing k the entrepreneur is better off. Thus, implementing $\bar{k}$ is incentive compatible if and only if both conditions in items 1 and 2 hold. Once we established that incentive compatibility does not change by increasing the set of technologies, Proposition 1 follows. □

Proof of Proposition 7 . Without loss of generality let us assume that $\mathbf{1}_{\tau_{1k}}$ occurs before $\mathbf{1}_{\tau_{2k}}$, as we can always rename τ_{1k} and τ_{2k}. The incentive compatibility proof is slightly adjusted to accommodate the increments. Suppose the entrepreneur follows a strategy $k \neq \bar{k}$ up to time t and after time t moves to $\bar{k}$. The total expected payoff from following this strategy is

$$X_t = \int_0^t e^{-\gamma s} \Big((1-q)\, dI_s + q dY_s \Big) + e^{-\gamma t} W_t.$$

By Itô's Lemma

$$\begin{aligned}
e^{\gamma t} dX_t &= (1-q)\, dI_t + q\,(\mu_k + \alpha_k \mathbf{1}_{\tau_k})\, dt + q\,(dY_t - (\mu_k + \alpha_k \mathbf{1}_{\tau_k})\, dt) \\
&\quad - \gamma W_t dt + \gamma W_t dt - (1-q)\, dI_t - q \left(\mu_{\bar{k}} + \alpha_{\bar{k}} \mathbf{1}_{\tau_{\bar{k}}}\right) dt + (\beta_t - q) \left(dY_t - \mu_{\bar{k}} dt - \alpha_{\bar{k}} \mathbf{1}_{\tau_{\bar{k}}} dt\right) \\
&= q\,(\mu_k + \alpha_k \mathbf{1}_{\tau_k})\, dt + q\,(dY_t - (\mu_k + \alpha_k \mathbf{1}_{\tau_k})\, dt) - q \left(\mu_{\bar{k}} + \alpha_{\bar{k}} \mathbf{1}_{\tau_{\bar{k}}}\right) dt \\
&\quad + (\beta_t - q) \Big[(\mu_k + \alpha_k \mathbf{1}_{\tau_k}) - \left(\mu_{\bar{k}} + \alpha_{\bar{k}} \mathbf{1}_{\tau_{\bar{k}}}\right) \Big] dt + (\beta_t - q)\,(dY_t - (\mu_k + \alpha_k \mathbf{1}_{\tau_k})\, dt) \\
&= \beta_t \Big[(\mu_k + \alpha_{1k} \mathbf{1}_{\tau_{1k}} + \alpha_{2k} \mathbf{1}_{\tau_{2k}}) - \left(\mu_{\bar{k}} + \alpha_{1\bar{k}} \mathbf{1}_{\tau_{1\bar{k}}} + \alpha_{2\bar{k}} \mathbf{1}_{\tau_{2\bar{k}}}\right) \Big] dt + \beta_t\,(dY_t - (\mu_k + \alpha_k \mathbf{1}_{\tau_k})\, dt)
\end{aligned}$$

Suppose $\mathbf{1}_{\tau_{1\bar{k}}} = \mathbf{1}_{\tau_{2\bar{k}}} = 1$ does *not* hold (item 2): If $\beta_t \leq 0$ the value process X_t has a non-positive drift,

$$\beta_t \Big[(\mu_k + \alpha_{1k} \mathbf{1}_{\tau_{1k}} + \alpha_{2k} \mathbf{1}_{\tau_{2k}}) - \left(\mu_{\bar{k}} + \alpha_{1\bar{k}} \mathbf{1}_{\tau_{1\bar{k}}} + \alpha_{2\bar{k}} \mathbf{1}_{\tau_{2\bar{k}}}\right) \Big] \leq 0. \tag{A.12}$$

If there is a positive probability event that $\beta_t > 0$ for $t < T$, by implementing k the entrepreneur is better off.
$\mathbf{1}_{\tau_{1\bar{k}}} = \mathbf{1}_{\tau_{2\bar{k}}} = 1$ does hold (item 1): If $\beta_t \geq 0$ the value process X_t has a non-positive drift,

$$\beta_t \Big[(\mu_k + \alpha_{1k} \mathbf{1}_{\tau_{1k}} + \alpha_{2k} \mathbf{1}_{\tau_{2k}}) - (\mu_{\bar{k}} + \alpha_{1\bar{k}} + \alpha_{2\bar{k}}) \Big] \leq 0. \tag{A.13}$$

If there is a positive probability event that $\beta_t < 0$ for $t < T$, by implementing $k \neq \bar{k}, \in \mathcal{K}$ the entrepreneur is better off. Thus, implementing $\bar{k}$ is incentive compatible if and only if both conditions in items 1 and 2 hold. Once we established that incentive compatibility holds we follow the same principles as in the proof of Proposition 1. □

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