RISK MEASURES FOR INVESTMENT VALUES AND RETURNS BASED ON SKEWED-HEAVY TAILED DISTRIBUTIONS: ANALYTICAL DERIVATIONS AND COMPARISON

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November 2018

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ABSTRACT

The skewed generalized t (SGT) displays an exceptional ability in modelling the tails of the empirical distributions of returns of financial and other assets. This feature makes it an appealing candidate for the computation of value at risk and expected shortfall measures, used by regulators, investors, portfolio managers and actuaries to measure and manage the risk exposure of their assets. This paper makes a specific contribution by deriving the analytical equations for the computation of value at risk, expected shortfall and downside risk measures for asset values and returns based on the SGT distribution. An assessment using simulations and estimation show that risk measures based on returns overestimate risk exposure.

Keywords: downside risk, expected shortfall, flexible distributions, risk management, SGT, value at risk

JEL: C46, G10, G20

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1. Introduction

Following the savings and loans association (S&L) crisis in the U.S. in the 1980's, there has been an increased interest and focus on the assessment of risk exposure of financial institutions, including banks and insurance companies. Due to the recent financial crisis in many European Union countries, the interest in risk measures used by regulators to set capital requirements for financial institutions as protection against catastrophic events has re-emerged.

Measurement of investment risk is also an important issue for investors, portfolio managers and actuaries. The typical risk measures employed are value at risk (VaR), which gives the maximum loss of an investment during a fixed period when the worse outcomes with a small chance of occurrence are excluded and expected shortfall (ES), which gives the conditional expected value of the investment loss exceeding the VaR value. Actuaries and insurance practitioners use VaR and ES measures to set capital reserves and insurance deductibles premia; e.g., Cummins et al. (1990). Another important risk measure is downside volatility, measured by the variance of returns below the mode or some other threshold, e.g., Ang et al. (2006), Danielson et al. (2006) and Feunou et al. (2012). [1]

Flexible probability distributions accounting for extreme values, skewness and fat tails, such as, the generalized Pareto, the generalized extreme value (GEV), the Box-Cox-GEV of Bali (2003), the inverse hyperbolic sine (IHS) of Johnson (1949), the exponential generalized beta of the second kind (EGB2) of McDonald and Xu (1995), the skewed t of Hansen (1994), the generalized t of McDonald and Newey (1988), the skewed generalized-t (SGT) of Theodossiou (1998) and its special case the skewed generalized

[1] An excellent survey of recent methodological and empirical developments in VaR forecasting and testing can be found in Nieto and Ruiz (2016). A detailed comparison of VaR Methodologies can be found in Abad and Benito (2013). A proficient review of known methods in estimating expected shortfall measures with emphasis on recent developments is presented in Nadarajah et al. (2014).

error distribution (SGED), are often employed in the computation of VaR and ES measures, associated with extreme downside periods. Bali and Theodossiou (2008) using the coverage tests of Kupiec (1995) and Christoffersen (1998) evaluated the risk measurement performance of these distributions and found that the SGT, EGB2 and IHS performed as well as the more specialized extreme value distributions in modelling the tail behaviour of portfolio returns.

Because of its exceptional ability to model the tails of the empirical distributions of returns of assets, such as, stocks, currencies, precious metals and commodities, the SGT entails closer attention.[2] Another interesting feature, is its nesting of many popular distributions often employed in theoretical and empirical research in economics, finance and management. The set includes the symmetric and skewed versions of the student's t, Cauchy, Laplace, normal and generalized error distribution.

This paper makes a specific contribution by deriving, for the first time, analytical SGT and SGED equations for the computation of ES and downside risk measures for asset values and returns.[3] Due to the absence of such equations ES measures were computed using numerical methods or approximation formulas. Considering that both the VaR and ES are tail measures at the 0.5% or 1% probability, the derived equations are expected to improve their computational accuracy. In the extant literature on risk measurement, the focus has been on risk measures for returns. Note that VaR measures for investment values can be computed directly using VaR measures for returns. This is not, however, the case for ES and downside risk measures for asset values. The latter measures are functions of the higher lower partial

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[2] The SGT has been used extensively in research on volatility, estimation of empirical distributions of returns, computation of risk measures, robust estimation, estimation of asset pricing models and pricing of options. Moreover, it is incorporated in programs such as Gauss, for maximum likelihood estimation. Two notable articles using a variation of the Hansen's (1994) skewed student t, called the generalized asymmetric student t, in computing VaR and ES measures are those Zhu and Galbraith (2010) and (2011).

[3] The SGED was used, among others, by Allen and Bali (2007) to assess the operational risk of financial institutions, Cheng and Hung (2011) to measure the risk of holding futures on petroleum and metals and Lyu et al. (2017) to compute VaR measures for crude oil returns.

moments of returns. Last, but not least, the derived analytical equations can be easily incorporated into computer programs for more accurate computation and easier implementation of risk measures.

Sections 2 and 3 present the analytical equations for the cumulative probability, the quantile value and the moments of the SGT and SGED, respectively. Section 4 presents the analytical equations for the computation of VaR, ES and downside risk measures for asset values and returns. Section 5 presents the simulation and estimation results. The paper ends with a summary and conclusions.

2. Skewed Generalized t

Under the SGT, the probability mass function for returns is

$$dF_y = \frac{k}{2\phi} n^{-\frac{1}{k}} B\left(\frac{1}{k}, \frac{n}{k}\right)^{-1} \left(1 + \frac{1}{n} \frac{|y-m|^k}{\left(1+sgn(y-m)\lambda\right)^k \phi^k}\right)^{-\frac{n+1}{k}} dy \ , \tag{1}$$

where m is the mode, ϕ is a scaling constant related to the standard deviation of y, λ is an asymmetry parameter, k and n are tail parameters and sgn is the sign function, i.e., $sgn(y-m) = -1$, for $y < m$ and $sgn(y - m) = 1$, for $y > m$ and $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}\, dt$ is the beta function.[4]

Using the transformation

$$w = \frac{y-m}{\left(1+sgn(y-m)\lambda\right)\phi} \ , \tag{2}$$

the SGT can be re-written as

$$dF_y = \left(1+sgn(w)\lambda\right) dF_w \ , \tag{3}$$

where

$$dF_w = \frac{k}{2} n^{-\frac{1}{k}} B\left(\frac{1}{k}, \frac{n}{k}\right)^{-1} \left(1 + \frac{1}{n}|w|^k\right)^{-\frac{n+1}{k}} dw \ .^{5} \tag{4}$$

The above decomposition aids greatly the derivation of the various results in this paper.

[4] Several equivalent SGT specifications appeared the literature. The specification in this paper, which is easier to use in the derivations, is that of Savva and Theodossiou (2018).

[5] Equation (3) is obtained via the substitution $y = m + \left(1+sgn(w)\lambda\right)\phi w$ and $dy = \left(1+sgn(w)\lambda\right)\phi\, dw$ into equation (1). Note that it is obvious from equation (2) that $sgn(y-m) = sgn(w)$.

The SGT cumulative distribution function is

$$q \equiv P\left(y \le y_q\right) = \int_{-\infty}^{y_q} dF_y = \frac{1}{2}(1-\lambda) + \frac{1}{2} sgn\left(w_q\right)\left(1 + sgn\left(w_q\right)\lambda\right) \mathrm{IB}\left(t_q; \frac{1}{k}, \frac{n}{k}\right) , \tag{5}$$

where $y_q \in R, w_q = \left(y_q - m\right) \Big/ \left(1 + sgn\left(y_q - m\right)\lambda\right)\phi, \quad t_q = 1 \Big/ \left(1 + n\left|w_q\right|^{-k}\right)$ and $\mathrm{IB}(\cdot)$ is the incomplete beta function ratio. Given the probability q, the quantile value of y can be obtained using the equations

$$y_q = m + \left(1 + sgn\left(w_q\right)\lambda\right)\phi w_q \ , \tag{6}$$

where

$$w_q = sgn\left(q - (1-\lambda)/2\right) n^{\frac{1}{k}}\, t_q^{\frac{1}{k}} \left(1 - t_q\right)^{-\frac{1}{k}} \ , \tag{7}$$

$$t_q = \mathrm{IB}^{-1}\left(\frac{2\left|q - (1-\lambda)/2\right|}{\left(1 + sgn\left(q - (1-\lambda)/2\right)\lambda\right)}; \frac{1}{k}, \frac{n}{k}\right) \tag{8}$$

and $\mathrm{IB}^{-1}(\cdot)$ is the inverse incomplete beta function ratio; see Appendix 1, results 4 and 5, for the derivations.

The j^{th} lower partial moment function of excess returns to mode, $y - m$, is

$$M_j^-\left(y_q\right) \equiv E\left(\left(y-m\right)^j \middle| y \le y_q\right) = \frac{1}{P\left(y \le y_q\right)} \int_{-\infty}^{y_q} \left(y-m\right)^j dF_y$$

$$= \frac{1}{2q}\left[\left(-1\right)^j \left(1-\lambda\right)^{j+1} + \left(sgn\left(w_q\right)\right)^{j+1} \left(1 + sgn\left(w_q\right)\lambda\right)^{j+1} \mathrm{IB}\left(t_q; \frac{j+1}{k}, \frac{n-j}{k}\right)\right] G_j \phi^j \ , \tag{9}$$

where

$$G_j = n^{\frac{j}{k}} \mathrm{B}\left(\frac{j+1}{k}, \frac{n-j}{k}\right) \mathrm{B}\left(\frac{1}{k}, \frac{n}{k}\right)^{-1} \ . \tag{10}$$

For the derivations of the above integral, see Appendix 1, result 3. The moment function for y is obtained as a limiting case of the above lower partial moment equation as $y_q \to \infty$, [6]

$$M_j \equiv E\left(y-m\right)^j = M_j^-(\infty) = \frac{1}{2}\left[\left(-1\right)^j \left(1-\lambda\right)^{j+1} + \left(1+\lambda\right)^{j+1}\right] G_j \phi^j . \tag{11}$$

Using M_j, the mean and variance of y are found to be

$$\mu = m + M_1 = m + 2\lambda G_1 \phi \tag{12}$$

and

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[6] As $y_q \to \infty,\ w_q \to \infty,\ sgn(w_q) = 1,\ t_q = 1,\ y_q \to \infty,\ w_q \to \infty,\ sgn(w_q) = 1,\ t_q = 1,$ The moments exist for $j < n$.

$$\sigma^2 = M_2 - M_1^2 = \left(\left(1+3\lambda^2\right)G_2 - 4\lambda^2 G_1^2\right)\phi^2 \ . \tag{13}$$

3. Skewed Generalized Error Distribution

The Skewed Generalised Error Distribution (SGED) is obtained as a limiting case of the SGT as $n \to \infty$. Under the SGED, the probability mass function for returns is [7]

$$dF_y = \frac{1}{2\phi} k^{1-\frac{1}{k}} \Gamma\left(\frac{1}{k}\right)^{-1} \exp\left(-\frac{1}{k}\frac{|y-m|^k}{\left(1+sgn\left(y-m\right)\lambda\right)^k \phi^k}\right) dy \ , \tag{14}$$

where m is the mode, ϕ is a scaling constant related to the standard deviation of y, λ is an asymmetry parameter, k is a tail parameter, sgn is the sign function and $\Gamma\left(a\right) = \int_0^\infty t^{a-1}e^{-t}dt$ is the gamma function. Its cumulative distribution function is

$$q \equiv P\left(y \le y_q\right) = \frac{1}{2}(1-\lambda) + \frac{1}{2} sgn\left(w_q\right)\left(1+sgn\left(w_q\right)\lambda\right) \mathrm{I}\Gamma\left(t_q ; \frac{1}{k}\right) \ , \tag{15}$$

where $y_q \in R, w_q = \left(y_q - m\right) \Big/ \left(1+sgn\left(y_q-m\right)\lambda\right)\phi,\ t_q = k^{-1}\left|w_q\right|^k$ and $\mathrm{I}\Gamma(\cdot)$ is the incomplete gamma function ratio. Given the probability q, the quantile value of y can be computed using the equations

$$y_q = m + \left(1+sgn\left(w_q\right)\lambda\right)\phi w_q \ , \tag{16}$$

where

$$w_q = sgn\left(q-(1-\lambda)/2\right) k^{\frac{1}{k}}\ t_q^{\frac{1}{k}} \ , \tag{17}$$

$$t_q = \mathrm{I}\Gamma^{-1}\left(\frac{2\left|q-(1-\lambda)/2\right|}{\left(1+sgn\left(q-(1-\lambda)/2\right)\lambda\right)} ; \frac{1}{k}\right) \tag{18}$$

and $\mathrm{I}\Gamma^{-1}(\cdot)$ is the inverse of the incomplete gamma function ratio; see Appendix 1, result 4 and 5, for the derivations of equations (15) and (16). The j^{th} lower partial moment function of excess returns to mode,

$$M_j^-\left(y_q\right) \equiv E\left(\left(y-m\right)^j \middle| y \le y_q\right) = \frac{1}{P\left(y \le y_q\right)} \int_{-\infty}^{y_q} \left(y-m\right)^j dF_y$$

$$= \frac{1}{2q}\left[\left(-1\right)^j \left(1-\lambda\right)^{j+1} + \left(sgn\left(w_q\right)\right)^{j+1} \left(1+sgn\left(w_q\right)\lambda\right)^{j+1} \mathrm{I}\Gamma\left(t_q ; \frac{j+1}{k}\right)\right] G_j \phi^j , \tag{19}$$

where

$$G_j = k^{\frac{j}{k}} \Gamma\left(\frac{j+1}{k}\right) \Gamma\left(\frac{1}{k}\right)^{-1} \ ; \tag{20}$$

[7] This specification is from Theodossiou (2015) and differs slightly from that in Theodossiou (2000).

see Appendix 1, result 3, for the derivations. The equations for M_j, μ and σ^2 are identical to those of the SGT and, therefore, are not presented; see equations (11) – (13).

4. Risk Measures

The end of the period value of an investment is

$$V_1 = V_0 e^{y} \ , \tag{21}$$

where V_0 is the initial investment and y is the return for the period. The following expansion around the mode m provides the basis for the computation of the moments of the investment values

$$V_1^r = V_0^r e^{ry} = V_0^r e^{rm} e^{r(y-m)} \approx V_0^r e^{rm} \sum_{j=1}^{s} \frac{r^j}{j!} (y-m)^j \ , \text{ for } r = 1, 2, \ldots \tag{22}$$

Value at Risk

The Value at Risk (VaR) is a measure of the maximum investment loss during a fixed period when the worse outcomes with a small probability of occurrence are excluded. The period can be a day, a week or larger. The probability of worse outcomes, denoted by q, is usually set to 0.005 or 0.01. The VaR for the investment is defined as the difference between the initial investment V_0 and the q-quantile V_q. That is,

$$VaRV_q = V_0 - V_q = V_0\left(1 - e^{y_q}\right) \approx -y_q V_0 = VaR_q V_0 \ . \tag{23}$$

Often, the VaR measure is reported as a percent of the initial investment V_0, i.e.,

$$VaR_q = \left(V_0 - V_q\right) \big/ V_0 \approx -y_q. \tag{24}$$

The quantile for investment values is computed using the equation [8]

$$V_q = V_0 e^{y_q}, \tag{25}$$

where

$$y_q = m + (1-\lambda)\phi w_q. \tag{26}$$

Because q is small, $sgn(q - (1 - \lambda) / 2) = -1$. Equations (7) and (8) for the SGT simplify to

$$w_q = -n^{\frac{1}{k}} \, t_q^{\frac{1}{k}} \left(1 - t_q\right)^{-\frac{1}{k}} \tag{27}$$

and

$$t_q = \mathrm{IB}^{-1}\left(2\left|q - (1-\lambda)/2\right| \big/ (1-\lambda); 1/k, n/k\right). \tag{28}$$

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[8] V_q satisfies $P(V_1 \le V_q) = P(y \le y_q) = q.$ The return quantile is obtained through the inversion $y_q = P^{-1}(q)$.

Equations (17) and (18) for the SGED simplify to

$$w_q = -k^{\frac{1}{k}}\ t_q^{\frac{1}{k}} \tag{29}$$

and

$$t_q = \mathrm{I\Gamma}^{-1}\left(2\left|q-(1-\lambda)/2\right|/(1-\lambda);1/k\right). \tag{30}$$

Expected shortfall

The expected shortfall is defined as the expected value of the investment loss exceeding the value at risk or $y < y_q < 0$. It represents the expected value of the worst losses, i.e., losses larger than $V_0 - V_q$ or $-y_q$.

The expected shortfall equations of returns for the SGT and SGED are respectively

$$ES_q = -E\left(y\middle| y \le y_q\right) = -m - M_1^-\left(y_q\right) = -m + \frac{(1-\lambda)^2}{2q}\left[1-\mathrm{IB}\left(t_q;\frac{2}{k},\frac{n-1}{k}\right)\right]G_1\phi \tag{31}$$

and

$$ES_q = -E\left(y\middle| y \le y_q\right) = -m - M_1^-\left(y_q\right) = -m + \frac{(1-\lambda)^2}{2q}\left[1-\mathrm{I\Gamma}\left(t_q;\frac{2}{k}\right)\right]G_1\phi\ . \tag{32}$$

The terms G_1 and t_q are computed using equations (10) and (28) for the SGT and equations (20) and (30) for the SGED.[9] Note that ES_q depends on the first lower partial moment of excess returns, only.

The equation for expected shortfall for investment values is

$$ESV_q = E\left(V_0 - V_1\middle|V_1 \le V_q\right) = V_0 - V_0 e^m E\left(e^{y-m}\middle|y \le y_q\right)$$

$$\approx V_0 - V_0 e^m \sum_{j=0}^{s}\frac{1}{j!}E\left((y-m)^j\middle|y \le y_q\right) = V_0 - V_0 e^m \sum_{j=0}^{s}\frac{1}{j!}M_j^-\left(y_q\right)\ . \tag{33}$$

It is a function of the lower partial moments of returns, computed using equation (9) for the SGT and equation (19) for the SGED. As such, it cannot be computed using ES_q, as in the case of VaR measures. ESV_q can be approximated quite accurately using the first three to four partial moments, i.e., $s = 3$ or 4.

Downside Risk

Returns below a set threshold are generally unfavourable. Following Feunou et al. (2012), the mode, being the maximum likelihood point of the distribution, is used as the threshold to compute downside risk. The j^{th} lower partial moment function for downside excess returns to mode is

[9] For $y_q < 0$, $sgn(w_q) = -1$ and $j=1$, eq. (9) reduces to $M_1^-\left(y_q\right) = -(2q)^{-1}(1-\lambda)^2\left[1-\mathrm{IB}\left(t_q;2/k,(n-1)/k\right)\right]G_1\phi$ and eq. (19) to $M_1^-\left(y_q\right) = -(2q)^{-1}(1-\lambda)^2\left[1-\mathrm{I\Gamma}\left(t_q;2/k\right)\right]G_1\phi$.

$$M_j^-(m) \equiv E\left((y-m)^j \middle| y \le m\right) = \frac{1}{P(y \le m)} \int_{-\infty}^{m} (y-m)^j \, dF_y$$

$$= (-1)^j (1-\lambda)^j \phi^j \, 2\int_0^{\infty} w^j dF_w = (-1)^j (1-\lambda)^j G_j \phi^j, \text{ for } j = 1, 2, \ldots \tag{34}$$

Note that $y - m = (1 + sgn(w)\lambda)\phi w$ and $P(y \le m) = (1-\lambda)/2$.

Likewise, the lower partial moment function for downside investment values, is

$$MV_r^-(V_m) \equiv E\left(V_1^r \middle| V_1 \le V_m\right) = V_0^r e^{rm} E\left(e^{r(y-m)} \middle| y \le m\right)$$

$$\approx V_0^r e^{rm} \sum_{j=0}^{s} \frac{r^j}{j!} E\left((y-m)^j \middle| y \le m\right) = V_0^r e^{rm} \sum_{j=0}^{s} \frac{r^j}{j!} M_j^-(m), \text{ for } r = 1, 2, \ldots \tag{35}$$

Note that $V_m \equiv V_0\, e^m$. It follows easily from the above, that the equations for the computation of downside risk for returns and investment values are, respectively,

$$\mathrm{var}\left(y \middle| y \le m\right) = M_2^-(m) - \left(M_1^-(m)\right)^2 = (1-\lambda)^2 \left(G_2 - G_1^2\right)\phi^2 \tag{36}$$

and

$$\mathrm{var}\left(V_1 \middle| V_1 < V_m\right) = MV_2^-(V_m) - \left(MV_1^-(V_m)\right)^2 \approx V_0^2 e^{2m} \left[\sum_{j=0}^{s} \frac{2^j}{j!} M_j^-(m) - \left(\sum_{j=0}^{s} \frac{1}{j!} M_j^-(m)\right)^2\right]. \tag{37}$$

Similarly, the downside risk for values depends on the higher moments of excess returns to mode.

5. Assessment of Risk Measures

Research has focused on the computation of risk measures for returns. In the case of VaR, the risk exposure of an investment, in monetary units (VaR of investment value), is

$$VaRV_q = V_0 - V_q = V_0(1 - e^{y_q}) \approx -y_q V_0 = VaR_q V_0,$$

where V_0 is the current value of the investment and $-y_q$ is the VaR of returns; see equations (23) and (24).

In the case of expected shortfall, the per monetary unit risk exposure of the investment

$$ESV_q / V_0 = 1 - E\left(e^y \middle| y \le y_q\right).$$

depends on all lower partial moments, $M_j^-(y_q)$, for j = 1, 2, …; see equation (33). The ES for returns

$$ES_q = -E\left(y \middle| y \le y_q\right),$$

depends on $M_1^-\left(y_q\right)$, only; see equations (31) and (32). The relative bias between the two measures is

$$ES_q - \text{bias} = 100 \times \frac{V_0 ES_q - ESV_q}{ESV_q}\,. \tag{38}$$

Figure 1 presents a 3-D diagram of the above bias for values of n = 5, 10 and ∞, the asymmetric parameter $-0.5 \le \lambda \le 0.5$ and the kurtosis parameter $1 \le k \le 2$. In all cases the bias is positive indicating that the ES_q overestimates risk exposure. The relative bias ranges between 1% and 6%. The largest relative bias occurs for k = 1 and λ = –0.5 and the smaller for k = 2 and λ = 0.5. In general, smaller values of the asymmetry parameter λ and / or smaller values of the tail parameters k and n result in larger biases.[10]

Downside risk for returns is measured by the standard deviation of investment values and returns lower than the mode values of $V_m = V_0\, e^m$ and m, respectively. The relative bias for the two measures is

$$\text{Downside Risk} - \text{bias} = 100 \times \frac{V_0\sqrt{\text{var}\left(y \middle| y \le m\right)} - \sqrt{\text{var}\left(V_1 \middle| V_1 \le V_m\right)}}{\sqrt{\text{var}\left(V_1 \middle| V_1 \le V_m\right)}}\,. \tag{39}$$

Figure 2 presents a 3-D diagram of the downside risk bias for the same set of values for k, n and λ. As in the case, of ES_q, the bias is positive indicating that downside risk for returns overestimates the risk exposure of an investment. The bias ranges between 1.8% and 3.7%. The largest bias occurs for k = 1 and λ = –0.5 and the smaller bias for k = 2 and λ = 0.5. In general, smaller values for the asymmetry parameter λ and / or smaller values of the tail parameters k and n result in larger biases.

Table 1 presents estimates of the of the distribution of returns and the risk measures for investment values and returns for the IBM, Boeing, S&P500, GBP/USD and JY/USD using daily, weekly and monthly returns. Note that the ES-bias ranges between 3.5 and 30 basis points at the daily frequency, 10.5 and 164 basis points at the weekly frequency and 42 and 1,740 basis points at the monthly frequency. Clearly, these deviations are quite large. The downside risk measures follow a similar but not as an extreme pattern.

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[10] Note that the SGED bias is represented by the 3-D curve labelled as $n = \infty$. The skewed Laplace by the values k =1 and $n = \infty$, the skewed normal k =2 and $n = \infty$ and skewed t by k = 2.

6. Summary and Conclusions

The skewed generalized t (SGT) distribution can model well the tails of the empirical distribution of returns of stocks, stock indices, currencies, precious metals and commodities. Interestingly, the SGT includes many popular distributions often employed in theoretical and empirical work, such as the symmetric and skewed versions of the Cauchy, student's t, Laplace, normal and generalized error distribution as special cases.

This paper makes a specific contribution by deriving the analytical equations for the computation of value at risk (VaR), expected shortfall (ES) and downside risk measures for asset values and their returns for the SGT and its special case the SGED. Considering that both the VaR and ES are tail measures, the derived analytical equations provide an improvement to their computational accuracy.

Research on risk measurement has focused primarily on returns. Although, value at risk measures for asset values can be easily obtained from those of returns, this not the case for expected shortfall and downside risk measures for asset values. Simulation and estimation results depict that risk measures on returns overestimate risk exposure. The analytical equations developed in this paper, show that the latter measures also depend on the higher partial moments of the distribution of returns and as such need to be computed separately. These analytical equations can be easily incorporated into computer programs for more accurate computation an implementation of risk measures.

Appendix 1 Intermediate Results for the SGT and SGED

The equations below are used in the derivations of the SGT results.

$$1+(1/n)w^k=(1-t)^{-1} \text{ or } t=1/\left(1+nw^{-k}\right)$$

implies that $w=n^{\frac{1}{k}}t^{\frac{1}{k}}(1-t)^{-\frac{1}{k}}$, $dw=k^{-1}n^{\frac{1}{k}}t^{\frac{1}{k}-1}(1-t)^{-\frac{1}{k}-1}dt$ and $w^r dw=k^{-1}n^{\frac{r+1}{k}}t^{\frac{r+1}{k}-1}(1-t)^{-\frac{r+1}{k}-1}dt$.

The equations below are used in the derivations of the SGED results.

$$t=w^k/k,$$

implies that $w=k^{\frac{1}{k}}t^{\frac{1}{k}}$, $dw=k^{\frac{1}{k}-1}t^{\frac{1}{k}-1}dt$ and $w^r dw=k^{\frac{r+1}{k}-1}t^{\frac{r+1}{k}-1}dt$.

Result 1

SGT

$$\int_0^\infty w^r dF_w=\frac{1}{2}G_r=\frac{1}{2}n^{\frac{r}{k}}\mathrm{B}\left(\frac{r+1}{k},\frac{n-r}{k}\right)\mathrm{B}\left(\frac{1}{k},\frac{n}{k}\right)^{-1}.$$

SGED

$$\int_0^\infty w^r dF_w=\frac{1}{2}G_r=\frac{1}{2}k^{\frac{r}{k}}\Gamma\left(\frac{r+1}{k}\right)\Gamma\left(\frac{1}{k}\right)^{-1}.$$

Proof

SGT

$$\int_0^\infty w^r dF_w=\frac{k}{2}n^{-\frac{1}{k}}\mathrm{B}\left(\frac{1}{k},\frac{n}{k}\right)^{-1}\int_0^\infty w^r\left(1+(1/n)|w|^k\right)^{-\frac{n+1}{k}}dw$$

$$=\frac{1}{2}n^{\frac{r}{k}}\mathrm{B}\left(\frac{1}{k},\frac{n}{k}\right)^{-1}\int_0^1(1-t)^{\frac{n-r}{k}-1}t^{\frac{r+1}{k}-1}dt=\frac{1}{2}n^{\frac{r}{k}}\mathrm{B}\left(\frac{r+1}{k},\frac{n-r}{k}\right)\mathrm{B}\left(\frac{1}{k},\frac{n}{k}\right)^{-1}.$$

SGED,

$$\int_0^\infty w^r dF_w=\frac{1}{2}k^{1-\frac{1}{k}}\Gamma\left(\frac{1}{k}\right)^{-1}\int_0^\infty w^r e^{-|w|^k/k}dw$$

$$=\frac{1}{2}k^{\frac{r}{k}}\Gamma\left(\frac{1}{k}\right)^{-1}\int_0^\infty t^{\frac{r+1}{k}-1}e^{-t}dt=\frac{1}{2}k^{\frac{r}{k}}\Gamma\left(\frac{r+1}{k}\right)\Gamma\left(\frac{1}{k}\right)^{-1}.$$

Result 2

SGT

$$\int_0^{|w_q|}w^r dF_w=\frac{1}{2}\mathrm{IB}\left(t_q;\frac{r+1}{k},\frac{n-r}{k}\right)G_r, \text{ where } t_q=1/\left(1+n|w_q|^{-k}\right).$$

SGED

$$\int_0^{|w_q|} w^r\,dF_w = \frac{1}{2}\mathrm{I}\Gamma\left(t_q;\frac{r+1}{k}\right)G_r \text{ , where } t_q = |w_q|^k \big/ k.$$

Proof

SGT

$$\int_0^{|w_i|} w^r dF_w = \frac{k}{2}n^{-\frac{1}{k}}\mathrm{B}\left(\frac{1}{k},\frac{n}{k}\right)^{-1}\int_0^{|w_q|} w^r\left(1+(1/n)|w|^k\right)^{-\frac{n+1}{k}}dw = \frac{1}{2}n^{\frac{r}{k}}\mathrm{B}\left(\frac{1}{k},\frac{n}{k}\right)^{-1}\int_0^{t_q}(1-t)^{\frac{n-r}{k}-1}t^{\frac{r+1}{k}-1}dt$$

$$= \frac{1}{2}n^{\frac{r}{k}}\mathrm{B}\left(t_q;\frac{r+1}{k},\frac{n-r}{k}\right)\mathrm{B}\left(\frac{1}{k},\frac{n}{k}\right)^{-1} = \frac{1}{2}\mathrm{IB}\left(t_q;\frac{r+1}{k},\frac{n-r}{k}\right)G_r \text{ .}$$

SGED

$$\int_0^{|w_q|} w^r dF_w = \frac{1}{2}k^{1-\frac{1}{k}}\Gamma\left(\frac{1}{k}\right)^{-1}\int_0^{|w_q|} w^r e^{-|w|^k/k}dw = \frac{1}{2}k^{\frac{r}{k}}\Gamma\left(\frac{1}{k}\right)^{-1}\int_0^{t_q} t^{\frac{r+1}{k}-1}e^{-t}dt$$

$$= \frac{1}{2}k^{\frac{r}{k}}\Gamma\left(t_q;\frac{r+1}{k}\right)\Gamma\left(\frac{1}{k}\right)^{-1} = \frac{1}{2}\mathrm{I}\Gamma\left(t_q;\frac{r+1}{k}\right)G_r.$$

$\mathrm{IB}(x;a,b)$ and $\mathrm{I}\Gamma(x;a,b)$ are respectively the incomplete beta and incomplete gamma functions ratios

Result 3

SGT

$$\int_{-\infty}^{y_q}(y-m)^r\,dF_y = \frac{1}{2}\left[(-1)^r(1-\lambda)^{r+1} + \left(sgn(w_q)\right)^{r+1}\left(1+sgn(w_q)\lambda\right)^{r+1}\mathrm{IB}\left(t_q;\frac{r+1}{k},\frac{n-r}{k}\right)\right]G_r\phi^r \text{ ,}$$

$$\text{where } t_q = 1\Big/\left(1+n|w_q|^{-k}\right).$$

SGED

$$\int_{-\infty}^{y_q}(y-m)^r\,dF_y = \frac{1}{2}\left[(-1)^r(1-\lambda)^{r+1} + \left(sgn(w_q)\right)^{r+1}\left(1+sgn(w_q)\lambda\right)^{r+1}\mathrm{I}\Gamma\left(t_q;\frac{r+1}{k}\right)\right]G_r\phi^r \text{ ,}$$

$$\text{where } t_q = |w_q|^k \big/ k.$$

Proof

For $y_q \le m$, $w_q = (y_q-m)\Big/\left(\left(1+sgn(w_q)\lambda\right)\phi\right) < 0$,

$$\int_{-\infty}^{y_q}(y-m)^r\,dF_y = \phi^r\int_{-\infty}^{w_q} w^r\left(1+sgn(w)\lambda\right)^{r+1}dF_w = \phi^r\left[(1-\lambda)^{r+1}\int_{-\infty}^{0} w^r dF_w - (1-\lambda)^{r+1}\int_{w_q}^{0} w^r dF_w\right]$$

$$= \left[(-1)^r(1-\lambda)^{r+1}\int_0^{\infty} w^r dF_w - (-1)^r(1-\lambda)^{r+1}\int_0^{|w_q|} w^r dF_w\right]\phi^r.$$

For $y_i \ge m$, $w_i > 0$ and

$$\int_{-\infty}^{y_i} (y-m)^r \, dF_y = \phi^r \int_{-\infty}^{w_i} w^r \left(1+sgn(w)\lambda\right)^{r+1} dF_w = \phi^r \left[(1-\lambda)^{r+1} \int_{-\infty}^{0} w^r dF_w + (1+\lambda)^{r+1} \int_{0}^{w_i} w^r dF_w \right]$$

$$= \left[(-1)^r (1-\lambda)^{r+1} \int_0^{\infty} w^r dF_w + (1+\lambda)^{r+1} \int_0^{w_q} w^r dF_w \right] \phi^r .$$

Combining the above equations

$$\int_{-\infty}^{y_q} (y-m)^r \, dF_y = \frac{1}{2} \left[(-1)^r (1-\lambda)^{r+1} \int_0^{\infty} w^r dF_w + \left(sgn(w_i)\right)^{r+1} \left(1+sgn(w_i)\lambda\right)^{r+1} \int_0^{|w_q|} w^r dF_w \right] \phi^r$$

The substitution of the equations for $\int_0^{\infty} w^r dF_w$ and $\int_0^{|w_q|} w^r dF_w$ from results 1 and 2, gives for

SGT

$$\int_{-\infty}^{y_q} (y-m)^r \, dF_y = \frac{1}{2} \left[(-1)^r (1-\lambda)^{r+1} + \left(sgn(w_i)\right)^{r+1} \left(1+sgn(w_q)\lambda\right)^{r+1} \mathrm{IB}\left(t_q; \frac{r+1}{k}, \frac{n-r}{k}\right) \right] G_r \phi^r .$$

SGED

$$\int_{-\infty}^{y_q} (y-m)^r \, dF_y = \frac{1}{2} \left[(-1)^r (1-\lambda)^{r+1} + \left(sgn(w_q)\right)^{r+1} \left(1+sgn(w_q)\lambda\right)^{r+1} \mathrm{I\Gamma}\left(t_q; \frac{r+1}{k}\right) \right] G_r \phi^r .$$

Result 4

SGT

$$\int_{-\infty}^{y_i} dF_y = \frac{1}{2}(1-\lambda) + sgn(w_q)\frac{1}{2}\left(1+sgn(w_q)\lambda\right)\mathrm{IB}\left(t_q; \frac{1}{k}, \frac{n}{k}\right), \text{where } t_q = 1\Big/\left(1+n|w_q|^{-k}\right).$$

SGED

$$\int_{-\infty}^{y_q} dF_y = \frac{1}{2}(1-\lambda) + sgn(w_q)\frac{1}{2}\left(1+sgn(w_q)\lambda\right)\mathrm{I\Gamma}\left(t_q; \frac{1}{k}\right), \text{ where } t_q = |w_q|^k \big/ k .$$

Proof

The proof follows easily from result 3 by setting $r = 0$ and noting that $G_0 = 1$.

Result 5

The quantile value $y_q = m + \left(1+sgn(w_q)\lambda\right)\phi w_q$ can be computed using the following equations:

SGT

$$w_q = sgn\left(q-(1-\lambda)/2\right) n^{\frac{1}{k}} \, t_q^{\frac{1}{k}} \left(1-t_q\right)^{-\frac{1}{k}},$$

$$t_q = \mathrm{IB}^{-1}\left(\frac{2\left|q-(1-\lambda)/2\right|}{\left(1+sgn\left(q-(1-\lambda)/2\right)\lambda\right)}; \frac{1}{k}, \frac{n}{k} \right),$$

and IB^{-1} is the inverse of the incomplete beta function ratio.

SGED $$w_q = sgn\left(q-(1-\lambda)/2\right)k^{\frac{1}{k}}\ t_q^{\frac{1}{k}},$$

$$t_q = \text{I}\Gamma^{-1}\left(\frac{2\left|q-(1-\lambda)/2\right|}{\left(1+sgn\left(q-(1-\lambda)/2\right)\lambda\right)};\frac{1}{k}\right),$$

and $\text{I}\Gamma^{-1}$ is the inverse of the incomplete gamma function ratio.

Proof

SGT $$P\left(y\le y_q\right)=\int_{-\infty}^{y_q} dF_y = \frac{1}{2}(1-\lambda)+sgn\left(w_q\right)\frac{1}{2}\left(1+sgn\left(w_q\right)\lambda\right)\text{IB}\left(t_q;\frac{1}{k},\frac{n}{k}\right)=q.$$

The value of y_q corresponding to the probability value q is obtained from the inversion of the above cumulative probability distribution. That is

$$\text{IB}\left(t_q;\frac{1}{k},\frac{n}{k}\right)=\frac{2\left(q-(1-\lambda)/2\right)}{sgn\left(w_q\right)\left(1+sgn\left(w_q\right)\lambda\right)}=\frac{2\left|q-(1-\lambda)/2\right|}{\left(1+sgn\left(q-(1-\lambda)/2\right)\lambda\right)}.$$

Thus,

$$t_q = \text{IB}^{-1}\left(\frac{2\left|q-(1-\lambda)/2\right|}{\left(1+sgn\left(q-(1-\lambda)/2\right)\lambda\right)};\frac{1}{k},\frac{n}{k}\right).$$

The quantile value for y is given by

$$y_q = m+\left(1+sgn\left(w_q\right)\lambda\right)\phi w_q$$

where

$$w_q = sgn\left(q-(1-\lambda)/2\right)n^{\frac{1}{k}}\ t_q^{\frac{1}{k}}\left(1-t_q\right)^{-\frac{1}{k}}.$$

SGED $$P\left(y\le y_q\right)=\int_{-\infty}^{y_q} dF_y = \frac{1}{2}(1-\lambda)+sgn\left(w_q\right)\frac{1}{2}\left(1+sgn\left(w_q\right)\lambda\right)\text{I}\Gamma\left(t_q;\frac{1}{k}\right)=q.$$

The value of y_q corresponding to the probability value q is obtained from the inversion of the above cumulative probability distribution. That is

$$\text{I}\Gamma\left(t_q;\frac{1}{k}\right)=\frac{2\left(q-(1-\lambda)/2\right)}{sgn\left(w_q\right)\left(1+sgn\left(w_q\right)\lambda\right)}=\frac{2\left|q-(1-\lambda)/2\right|}{\left(1+sgn\left(q-(1-\lambda)/2\right)\lambda\right)}.$$

Thus,

$$t_q = \text{I}\Gamma^{-1}\left(\frac{2\left|q-(1-\lambda)/2\right|}{\left(1+sgn\left(q-(1-\lambda)/2\right)\lambda\right)};\frac{1}{k}\right).$$

The quantile value for y is given by

$$y_q = m + \left(1 + sgn\left(w_q\right)\lambda\right)\phi w_q$$

where

$$w_q = sgn\left(q - (1-\lambda)/2\right)k^{\frac{1}{k}}\ t_q^{\frac{1}{k}}.$$

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Table 1. Estimation of SGT parameters and Associated Risk Measures

A. Daily returns

Parameters	Boeing	IBM	S&P500	GBP/USD	JY/YSD
μ	0.0547	0.0279	0.0364	0.0044	0.0033
	(2.86)**	(1.77)	(4.12)**	(0.69)	(0.38)
σ	2.0015	1.729	1.0386	0.6055	0.6816
	(70.8)**	(48.56)**	(80.22)**	(107.29)**	(61.85)**
k	2.0564	2.153	1.2254	1.124	1.5781
	(25.63)**	(23.18)**	(28.88)**	(5.89)**	(16.02)**
n	3.6978	3.1575	6.3287	19.2201	5.9234
	(25.97)**	(24.9)**	(10.8)**	(0.63)	(6.11)**
λ	0.0222	0.0219	0.0061	0.0126	–0.0312
	(1.56)	(1.59)	(0.36)	(1.26)	(–1.86)
LR-Normal	5,065.9**	9,444.3**	3,069.6**	1,627.81**	810.5**
OBS	10,342	10,341	10,696	10,696	6,000
VaR_q	5.1494	4.4204	2.8694	1.6476	1.8732
VaRVq	5.0191	4.3241	2.8286	1.6341	1.8558
ESq	7.3258	6.6267	3.9061	2.0957	2.4823
ESVq	7.0231	6.376	3.8237	2.0729	2.4492
DRisk	1.403	1.2639	0.7582	0.4218	0.4901
DRiskV	1.3307	1.0889	0.7448	0.4181	0.4848

B. Weekly returns

Parameters	Boeing	IBM	S&P500	GBP/USD	JY/YSD
μ	0.225	0.1133	0.1444	0.0208	0.0125
	(2.27)*	(1.46)	(3.07)**	(0.7)	(0.29)
σ	4.7066	3.8463	2.2319	1.3781	1.5587
	(26.21)**	(24.34)**	(28.86)**	(45.84)**	(29.46)**
k	2.4393	2.2976	2.1933	1.8686	2.472
	(9.01)**	(10.01)**	(8.63)**	(9.13)**	(6.26)**
n	3.2337	3.0895	4.1117	6.2762	4.5274
	(11.28)**	(13.89)**	(5.78)**	(4.43)**	(4.97)**
λ	–0.0389	–0.027	–0.1066	0.0623	–0.1542
	(–1.04)	(–0.82)	(–3.45)**	(2.02)*	(–3.42)**
LR–Normal	883.0**	1,794.7**	277.3**	179.5**	132.3**
OBS	2,139	2,139	2,140	2,140	1,200
VaRq	12.3381	10.1018	6.112	3.3966	4.3253
VaRVq	11.6073	9.6083	5.929	3.3395	4.2331
ESq	18.4203	15.3648	8.5483	4.3403	5.8459
ESVq	16.78	14.5134	8.1433	4.2419	5.6605
DRisk	3.5504	2.9467	1.6967	0.8615	1.1804
DRiskV	2.4872	0.4364	1.6531	0.8449	1.1525

Table 1. (continued)

C. Monthly returns

Parameters	Boeing	IBM	S&P500	GBP/USD	JY/YSD
μ	0.4765	0.3962	0.6812	0.093	0.0353
	(1.09)	(1.06)	(3.46)**	(0.67)	(0.18)
σ	11.6402	9.464	4.2831	2.9692	3.1969
	(4.59)**	(4.91)**	(20.8)**	(22.78)**	(12.81)**
k	3.6056	3.4836	1.8668	1.7317	1.9878
	(2.64)**	(3.15)**	(4.29)**	(4.25)**	(3.01)**
n	2.6334	2.5224	7.013	9.0791	5.6107
	(5.01)**	(6.51)**	(1.5)	(1.14)	(1.21)
λ	–0.2681	–0.0509	–0.1619	0.0362	–0.0423
	(–3.38)**	(–0.56)	(–2.17)*	(0.57)	(–0.45)
LR–Normal	169.8**	347.5**	43.5**	24.9**	19.3**
OBS	492	492	492	492	276
VaRq	32.7993	22.7793	11.4198	7.35	8.4594
VaRVq	27.9632	20.3711	10.7919	7.0864	8.1115
ESq	55.5949	38.4683	14.8414	9.1523	11.0309
ESVq	28.1783	25.0175	13.7306	8.7291	10.4035
DRisk	10.8557	7.9097	3.2549	1.8821	2.2235
DRiskV	12.1094	8.3412	3.1019	1.8119	2.1253

Notes: Parentheses include the t-values of the estimators. **, * statistically significant at the 1% and 5% levels, respectively. Data for IBM, Boeing, S&P500 and GBP/USD cover the period 1/1/1976 to 30/12/2016 and for the JY/USD, 31/3/1994 to 30/12/2016. All risk measures are at computed at the 1% level. LR-Normal is a log-likelihood ratio test for normality. The remaining parameters are as defined in the text.

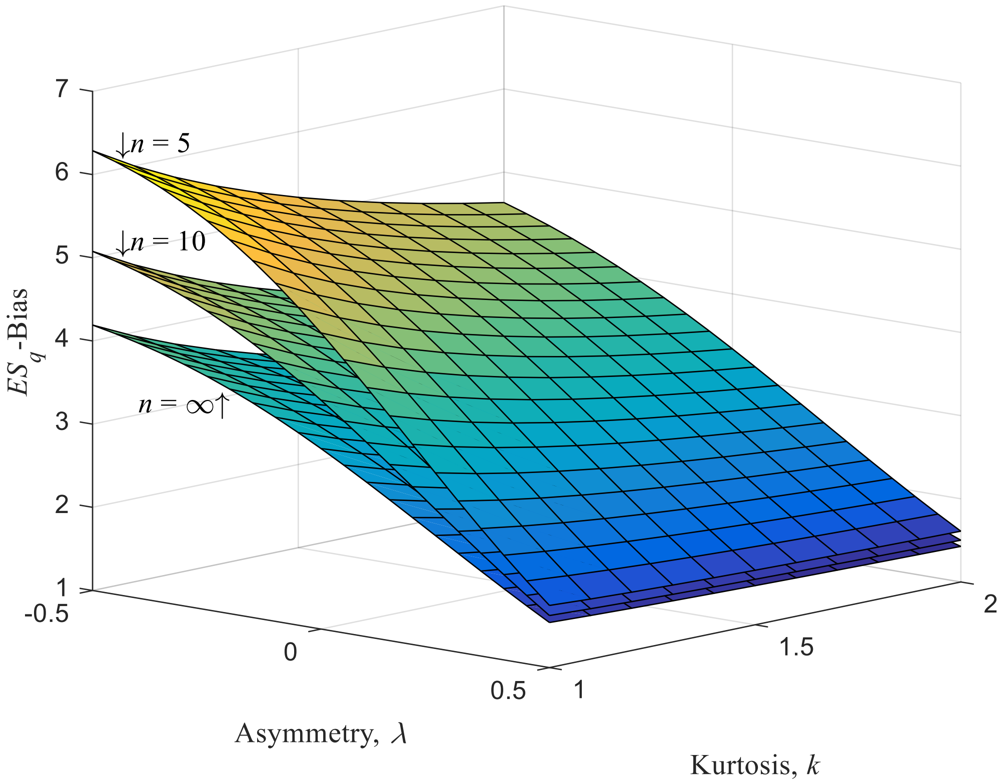

Figure 1. Expected Shortfall Bias

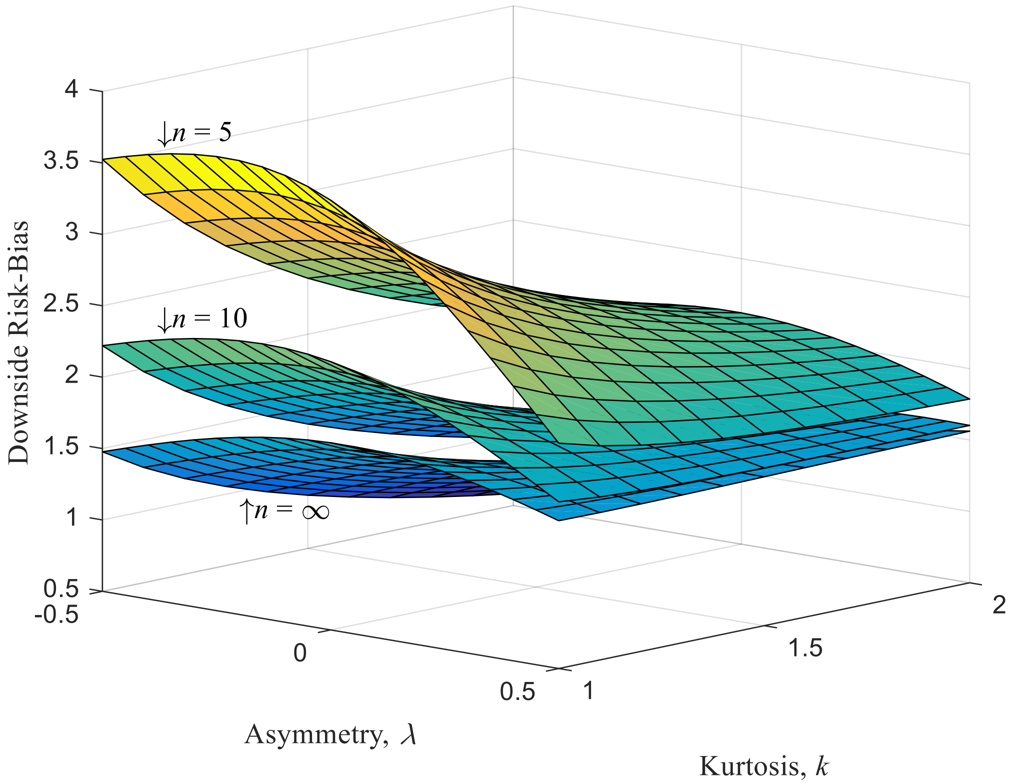

Figure 2. Downside Risk Bias